Monetary Policy with Heterogeneous Collateralized Borrowing

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Abstract

The purpose of this paper is to properly analyze the role of the value of collateral in a model of money, and its interaction with monetary policy in an environment with heterogeneous borrowing capacity. Asset pricing reflects the liquidity properties of the collateralizable asset. Away from the Friedman rule, interest rates are increasing in the expected return to collateral. Monetary policy acts to accommodate changes in the value of the asset used as collateral. Borrower’s heterogeneity plays a crucial role: asset pricing depends on the degree of heterogeneity and an increase in the liquidity premium of collateral goes hand in hand with lower interest rates.

JEL classification: E41; E44; E51; E52; E58, G12.

Keywords: collateral, liquidity, money, monetary policy, asset pricing.

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1 Introduction

Agents usually anticipate consumption by borrowing. In the last years the development of credit markets has walked hand by hand with a significant increase in consumer debt. Monacelli (2006) more specifically reports on the considerable degree of co-movement between total private consumption and household mortgage debt in the last fifteen years. Monacelli also reports on the significant increase in secured debt as a percentage of household debt.

On the other side of this remarkable credit extension, information and repayment issues have often arisen. Agents could use assets as collateral in order to alleviate those information and incentive problems. It is for good reason that secured debt has become a vastly common form of borrowing contract and collateral is widely required in most credit market transactions. There is a variety of assets that can serve as collateral, among which durable goods seem to be the predominant instrument. Nevertheless, these and other assets may often present features that could cause their price to be not so stable.

Bernanke and Gertler (1999, 2001) maintain that monetary policy should only respond to variability in asset prices insofar as they pose a threat to price stability or help forecast inflationary pressures. However, they reach that conclusion for a model that analyzes asset price bubbles in which the asset other than money is capital, as part of an investment portfolio and productive input. I will try to argue that things may be different when, in line with Iacoviello (2005) and Kiyotaki and Moore (1997), these assets are used as collateral and borrowing constraints are tied to the value of the asset. In fact, I derive a monetary policy scheme where, in equilibria where both money and collateral exist, the interest rate is related the aggregate value of collateral. Iacoviello (2005) is closer to Bernanke et al. (1996, 1999) where they describe a financial accelerator mechanism by which a shock to a firm’s net worth leads to a change in the value of its collateralizable assets, and this to an adjustment in production.

Instead, I here describe a world in which asset price movements may affect final consumption through collateral constraints. Ferraris and Watanabe (2008)

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1 According to Ferraris and Watanabe (2008) the value of all commercial and industrial loans intermediated by US banks and secured by collateral amount to 46.9% of the total value of loans in 2006. In the case of commercial loans, in 2004 up to 47.9% of US households were holding debt secured by their houses. In this sense, Iacoviello (2005) also reports on a large proportion of consumption borrowing secured by real estate.

2 Their price is supposed to be less volatile, since it is subject to longer term consumption or investment decisions.
present a model in which capital is used as collateral, but in principle its price does not play a role, and they concentrate on issues like optimal capital accumulation. The idea in this paper may be closer to that in Kiyotaki and Moore (1997). However, the asset that serves as collateral in their paper is also a factor of production, which affects the relation between asset prices, credit limits, and borrowing constraints in a very different way.

In this paper I build a model that both provides explicit micro foundations that make money essential and allows agents the possibility to store assets that may be used as collateral in a credit market. Even though the latter issue has been treated to a certain extent in the literature, the introduction of collateral in search-theoretic models of money is a very recent development, which counts only a handful of papers.\(^3\) In fact, this framework is the key mechanism that allows the model in this paper to deliver explicit asset pricing features that are not present in any model of collateral under different frameworks. More precisely, the asset pricing in these other papers is not able to reflect the liquidity properties of the assets used as collateral.

Furthermore, the value of the assets that serve as collateral plays a crucial role in this paper, for I model the value of collateral in a very natural way that allows me to analyze heterogeneity on the borrower’s side. To the best of my knowledge, this is another feature in the paper that has not been explicitly considered before. In other models with collateral and borrowing constraints (Bernanke and Gertler (1999), Iacoviello (2005), Kiyotaki and Moore (1997), and Monacelli (2006)) heterogeneity is introduced by means of giving agents different impatience rates. In this model, there will still be some agents who become lenders and other agents who become borrowers. However, only for a fraction of borrowers the value of the asset used as collateral will not be high enough to get an optimal amount of credit. Bernanke and Gertler (1995, 1999, 2001) assert that the condition of balance-sheets is determinant of agent’s ability to borrow and lend. In my model it is the value of the asset used as collateral what will determine how much an agent can borrow and whether an agent faces a borrowing limit or not. I believe that modeling such kind of heterogeneity is not only innovative but might be interesting for the analysis of monetary policy. More precisely, I argue that the presence of this kind of heterogeneity will have an impact on the profile of interest rates and the liquidity premium of the asset stored as collateral.

My work is in this sense related to that by Berentsen and Waller (2006). They study optimal policy where the monetary authority seeks to improve wel-

\(^3\) These papers will be discussed later.
fare by stabilizing short-run aggregate shocks. As a result they obtain that, away from the Friedman rule, the optimal policy is to smooth interest rates in order to smooth consumption. However, since their paper focuses on different questions, they do not model the use of collateral and their borrowers are homogeneous.

Finally, another relevant insight delivered by the model involves the asset pricing mechanism. Agents will value assets for not only their specific expected return, but also for the liquidity that they can provide in contingencies when used as collateral. Assets other than money are modeled in a way that they may yield less return than financial assets, but they possess better liquidity properties. On the other hand, these assets may be less liquid than fiat money but will bear an additional return, higher than that of mere cash. I discuss equilibria with money and a stored asset that has different properties than fiat money or financial assets, and that will still be valued for its “indirect” liquidity as a collateralizable asset. In fact, money and the asset that serves as collateral are rather complements in my model.

The rest of the paper is structured as follows. Section 2 describes the main aspects of the model. Section 3 discusses the features of the equilibrium and main results, and section 4 concludes.

2 The Model

The structure of the environment is similar to that in Berentsen et al. (2007), which itself builds on Lagos and Wright (2005). Time is discrete and there is a [0, 1] continuum of agents that live forever and discount future at rate $\beta \in (0, 1)$. In every period there are three markets that open sequentially in the following order: first, a credit market; second, a decentralized trade market where agents meet bilaterally; and finally a centralized Walrasian market. The commodities traded in the second and third market are perfectly divisible and of different types.

At the beginning of every period, and before entering the first market, uncertainty is resolved and agents receive a preference shock so that they find out both whether they are going to be producers or consumers in the second market of that period, and whether they will match with someone in that market.4 Producers will only produce and sell the good in the decentralized market to

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4 The second part of this assumption is usually related to information issues: agents would receive a signal that would tell them, with certain probability, whether they will match, and
agents who are consumers. Therefore, producers will not need money in the second market, as opposed to consumers who need it to buy the consumption good. Thus, at any period of time $t$ a credit market opens in which agents can reallocate their money balances before going into trade.

In this credit market agents (consumers or producers) who know they will not match may want to deposit their money in the bank and earn the interest at the nominal rate $i_t$. Consumers who know they will match want to borrow money to finance consumption in the next market. The net borrowing of a certain agent is denoted by $l_t$. When the credit market closes, trade takes place in a decentralized market where agents meet and bargain over the terms of trade in a bilateral and anonymous manner. Finally, in the third market agents work, consume, and readjust their portfolios for the next period.

In order to simplify the analysis and get a clear intuition of the functioning of the model I assume, without loss of generality, that loan and deposit contracts cannot be rolled over. As a consequence all financial contracts will have a duration of only one period. Thus, in the third market agents repay their loans or redeem their deposits, and decide the amount of money and collateral to be taken into next period.

Consumers in market 2 get utility $u(q)$ from consuming an amount $q > 0$ of the good, where $u'(q) > 0$, $u''(q) < 0$, $u'(0) = +\infty$, and $u'(\infty) = 0$. Producers, on the other hand, incur in a cost $c(q)$ for producing $q$ units of output in the second market, where $c'(q) > 0$, $c''(q) \geq 0$, $c'(0) = 0$. Furthermore, for the sake of simplicity I assume that $c(q) = q$. Agents in the centralized market get utility $U(x)$ from consumption with $U'(x) > 0$, $U''(x) \leq 0$, $U'(0) = \infty$, and $U'(+\infty) = 0$. Also, a technology is available in this market that allows agents to produce one unit of input with one unit of labor generating one unit of disutility.

so on. In order to avoid a whole set of different issues completely out of the scope of this paper, and to simplify the presentation and the analysis, such an assumption is made. With quasi-linear utility agents are happy to pay off all debt at the end of every period. While one-period debt contracts are very convenient, in equilibrium they turn out to be sufficient to cover the needs of any trading situation.

Lagos and Wright (2005) explain how these assumptions, along with quasi-linearity of preferences, lead to a degenerate distribution of money and asset holdings at the beginning of every period. In particular, they show how the assumptions regarding the utility functions allow to derive technical conditions to ensure that in equilibrium all agents produce and consume in the centralized market.
2.1 Money

A most important object in the model is (fiat) money, perfectly divisible and can be stored in any amount \( m_t \geq 0 \). Papers that model search frictions with money as a medium of exchange, from the early work of Kiyotaki and Wright (1989, 1993) to more modern models, provide a good description of those frictions that make money essential. The same basic assumptions are also established here. In particular, in the decentralized market there are some agents who want to consume but cannot produce, and others who can produce but do not desire to consume. Thus, trade in this market occurs in bilateral and anonymous meetings where trading histories are private information. Thus, we generate the need to use fiat money as a medium of exchange (Kocherlakota (1998), Wallace (2001)). Therefore, I am able to construct a model where money and credit coexist, which is the basic framework where collateral can be introduced and analyzed.

The per capita money stock in market 3 at any given period in time \( t \) is denoted by \( M_t \). The gross growth rate of money supply is given by \( \gamma = M_t / M_{t-1} \), where \( \gamma > 0 \). I consider a monetary authority that takes the interest rate, \( i \), as its policy variable. The money growth rate \( \gamma \) will then be determined so that it is compatible with the chosen level of \( i \). This approach makes it considerably easier to solve the model and facilitates comparisons with a majority of models where the interest rate is also the monetary policy variable. Changes in the monetary base take place through deterministic lump-sum injections or withdrawals of money, \( \tau M_t \), at the beginning of every period. Therefore, the net change in the aggregate money stock would be \( \tau M_{t-1} = (\gamma - 1)M_{t-1} \).

Henceforth, in order to keep notation simple, those variables corresponding to the next period will be indexed by +1, and those corresponding to the previous one will be indexed by \(-1\).

Let \( P \) denote the price of the good in the centralized market. Then, \( \phi = 1/P \) denotes the value of money, the real price of money in that market. I focus on equilibria where real allocations are constant through time. These will be referred to as stationary equilibria. In particular,

\[
\phi M = \phi_{-1} M_{-1} \equiv z .
\]

In these equilibria \( \phi_{-1}/\phi = P/P_{-1} = M/M_{-1} = \gamma \), which means that I am restricting my attention to equilibria where the growth rate of money supply \( \gamma \) is constant. In what follows, I will solve and study the model backwards to
better track the decisions of the agents.

## 2.2 Collateral

Typically, real assets that may be used as collateral include real estate, houses, or even capital. However, it is relatively often to collateralize assets that are more related to consumption goods, like all the securities based on credit card debts, car loans, or, as in with Berentsen and Monnet (2009), one can also think of low-risk low-yield assets. In this line, I assume that there exists a storage technology such that agents can store some amount of the general good in the centralized market, \( a_t \), that could be used as collateral in the next credit market. This technology will allow agents to recover \( \delta a_t \) in the next centralized market. The return \( \delta \) can be interpreted as an idiosyncratic shock. When the time comes to be on the credit market, each agent’s stored asset, \( a_t \), will have a different value, \( \delta a_t \). These shocks follow a certain distribution, \( F(\delta) \), about which only two assumptions are made: \( F(\delta) \) is continuous, and such that \( \delta \in [0, \hat{\delta}] \), where \( \hat{\delta} \geq 1 \) is the highest that the collateral of any given agent can be valued.\(^7\). Let \( \bar{\delta} \) be the mean of the distribution. I impose that \( \beta \bar{\delta} \leq 1 \); since when \( \beta \bar{\delta} > 1 \) agents would want to store an infinite amount of the good, which is inconsistent with equilibrium.\(^8\) This implicitly defines an upper bound for the value of collateral, \( \bar{\delta} \leq 1/\beta \).

This asset is special in the following sense: its return is specific to the agent that owns it. The idiosyncratic shock can be interpreted as a shock that endows the stored amount of good of each agent with specific characteristics so that different values are assigned to the asset across agents. Thus, in a Kiyotaki and Wright (1989) fashion, I assume that each agent likes the particular features of her asset but not those of the asset owned by any other agent. Thus, this asset cannot be traded in the decentralized market nor be used to secure trade credit between a producer and a consumer, which is another difference with a standard financial asset. However, whenever people find themselves in need for liquidity they can collateralize their assets.\(^9\)

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\(^7\) Berentsen et al. (2007) and Ferraris and Watanabe (2008) both model deterministic returns to the asset that is used as collateral.

\(^8\) If \( \beta \bar{\delta} = 1 \) agents may be indifferent between storing assets and not storing at all. In the following sections I discuss how the precise structure of the model generates the conditions under which only when \( \beta \bar{\delta} < 1 \) agents are willing to hold a finite amount of collateral.

\(^9\) As I already mentioned, Ferraris and Watanabe (2008) report that for those US households with home-secured debt their house was the guarantee of payment.
While this approach to model collateral broadly captures the essence of a real asset being stored for that purpose, it is very convenient to keep the model analytically tractable, and allows me to focus on the use of collateral in credit markets. Moreover, one important advantage of this approach is that the real asset presents characteristics different from those of financial assets and those of money, and that it is still valued both for its indirect liquidity as well as for its specific expected return.

2.3 First-best allocation

For any given stationary allocation \((q, a)\), where \(q\) is consumption in the decentralized market and \(a\) is asset holdings, the expected lifetime utility of a representative agent at the beginning of a period is

\[
(1 - \beta)W = U(x) - x + n \left[ \int_0^{\delta_c} u(q_c) + \int_{\delta_c}^{\tilde{\delta}} u(q_u) \right] - s \left[ \int_0^{\delta_c} q_c + \int_{\delta_c}^{\tilde{\delta}} q_u \right] + (\beta E[\delta] - 1) a.
\]

Notice that I differentiate both utility and production cost for constrained, \(u(q_c)\), and unconstrained borrowers, \(u(q_u)\). These two types are identified by a critical value, \(\delta_c\), which will be defined below. The first two terms represent the utility from consuming and producing in the centralized market. The third and fourth terms are the expected utility from consuming and producing the good in market 2. Finally, the last term reports the utility from producing collateral and receiving its return in the following period.

The first-best allocation, as would be chosen by a social planner, can be expressed by a vector \((x^*, q^*_j, a^*)\), \(j = c, u\), satisfying \(U'(x^*) = 1\), and \(q_j = q^*_j\), \(j = c, u\), where \(q^*\) solves \(u'(q) = 1\). Furthermore, \(a^* = 0\) if \(\beta E[\delta] < 1\), and \(a^*\) would be indeterminate if \(\beta E[\delta] = 1\). In other words, a social planner is never going to choose a positive amount of collateral when it is costly, unless there is an indirect return to it that makes it valuable, as will be shown below.

3 Equilibrium

I focus on symmetric stationary monetary equilibria where all agents of a given type behave in the same way and real money balances are constant through

\[\text{\footnotesize In order to ease notation, I do not write the term } dF(\delta) \text{ in the integrals.}\]
3.1 Third Market

At this stage agents get the return from having stored the real asset in the last period, $\delta a$, and decide how much to work, $h$, how much to consume, $x$, and how to readjust their portfolios for next period. That is, how much money, $m_{+1}$, and how much of the real asset, $a_{+1}$, to carry into next period. Finally, in order to settle financial claims, agents who were consumers in market 2 and borrowed $l$ units of money have to pay back $(1+i)l$ from their loan contracts. This amount is transferred to agents who were producers, or agents who did not match, and thus they redeem their deposits.

Given a distribution $F(\delta)$ and a policy arrangement $i$, let $V_3(m, l, a)$ be the expected lifetime utility of entering the third market with $m$ units of money, $l$ net borrowing, and $a$ units of the stored asset in period $t$. The problem of a representative agent in this market is

$$V_3(m, l, a) = \max_{x, h, m_{+1}, a_{+1}} \left[ U(x) - h + \beta V_1(m_{+1}, a_{+1}) \right]$$

subject to

$$x + \phi m_{+1} + a_{+1} = h + \phi m - \phi (1+i)l + \delta a.$$  

Substituting the budget constraint into (3) gives us

$$V_3(m, l, a) = \phi [m - (1+i)l] + \delta a + \max_{x, m_{+1}, a_{+1}} \left[ U(x) - x - \phi m_{+1} - a_{+1} + \beta V_1(m_{+1}, a_{+1}) \right].$$

The first order conditions are

$$U'(x^*) = 1$$  

$$\phi_{-1} \geq \beta V_1^m \quad (=0, \text{ if } m > 0)$$  

$$1 \geq \beta V_1^a \quad (=0, \text{ if } a > 0)$$

The superscript indicates the variable with respect to which the function is being differentiated.\(^{11}\) Thus, $V_1^m$ is the marginal value of taking an additional unit of money into the first market of period $t$, and equation (5) is the asset pricing equation. The envelope conditions are

$$V_3^m = \phi; V_3^l = -\phi (1+i); V_3^a = \delta.$$  

\(^{11}\)Note that equations (4) and (5) have been lagged one period for convenience of analysis.
As can be observed, the value function is linear. This, together with the assumptions on preferences and production costs listed in the description of the model, ensures that there are no wealth effects. Thus, a degenerate distribution of money and the asset is obtained such that all agents enter the following period with same amount of money and collateral.

3.2 Second Market

At the point of entering the second market there are three types of agents: among those who match there are agents who want to consume but cannot produce, *buyers* (*b*), and those who want to produce but do not wish to consume, *sellers* (*s*). Agents belonging to the third type, *others* (*o*), are those who do not match. They can be both producers or consumers. In case a buyer meets a seller, they bargain bilaterally. The terms of trade are determined in a simple bargaining procedure of take-it-or-leave-it offers made by the buyer.\textsuperscript{12} It can be easily shown that the equilibrium equations and main results do not vary under other pricing mechanisms, like competitive pricing.\textsuperscript{13}

An agent that walks into the second market with \( m \) units of money, net borrowing \( l \), and \( a \) units of collateral has lifetime utility \( V_{2j}(m_j, l_j, a) \), \( j = b, s, o \). Agents who do not match in the second market will just walk through it and into the third market with their asset and money holdings and their loans untouched. Thus, their value function is simply

\[
V_{2o}(m_o, l_o, a) = V_3(m_o, l_o, a) .
\]

The value functions for sellers and buyers who match are, respectively,

\[
\begin{align*}
V_{2s}(m_s, l_s, a) &= -c(q) + V_3(m_s + d, l_s, a), \\
V_{2b}(m_b, l_b, a) &= u(q) + V_3(m_b - d, l_b, a).
\end{align*}
\]

where \( m_j = m + l_j, j = b, s \). The term \( d \) refers the amount of cash paid for the quantity consumed. Thus, an agent who has consumed \( q \) units of the good in

\textsuperscript{12} Rocheteau and Wright (2005) also analyze competitive pricing. They show that, at least for a model of credit without collateral, some results may vary according to different pricing procedures. The quantity traded under bargaining may be lower than under competitive pricing due to the classic holdup problem also pointed out in Lagos and Wright (2005). However, in my model the quantity traded does not need to be lower since I consider take-it-or-leave-it offers by the buyer.

\textsuperscript{13} For a comprehensive analysis of several relevant market structures and their different results please refer to Rocheteau and Wright (2005).
market 2 enters market 3 with $m_b - d$ units of money. Similarly, an agent who has produced and sold $q$ units of the good in market 2 enters market 3 with $m_s + d$ units of money. The problem to be solved by the buyers looks like
\[
\begin{align*}
\max_{q,d} & \quad u(q) + V_3(m_b - d, l_b, a) - V_3(m_b, l_b, a) \\
\text{s.t.} & \quad -c(q) + V_3(m_s + d, l_s, a) - V_3(m_s, l_s, a) \geq 0 \\
& \quad d \leq m_b.
\end{align*}
\] (7)

Since I have assumed that $c(q) = q$, and the third market value functions are linear, this problem reduces to
\[
\begin{align*}
\max_{q,d} & \quad u(q) - \phi d \\
\text{s.t.} & \quad -q + \phi d \geq 0 \\
& \quad d \leq m_b.
\end{align*}
\] (8)

The solution to this problem is
\[
\begin{align*}
\text{if} \quad q^* \leq \phi m_b, \quad \text{then} & \quad \begin{cases} q = q^* \\ d = \frac{q^*}{\phi} \end{cases} \\
\text{if} \quad q^* > \phi m_b, \quad \text{then} & \quad \begin{cases} q = \hat{q} \\ d = m_b \end{cases}
\end{align*}
\] (9)

where $\hat{q}$ solves $c(q) = \phi m_b$. Given the maintained assumptions, this means that $\hat{q} = \phi m_b$. As will be shown below, in either case both the amount consumed, $q$, and the amount of money paid for it, $d$, depend only on $m$, which itself depends on $m$ and $a$. Particularly, $q = \phi m + \phi l_b = z + \phi l_b(a)$, where $z = \phi m$ are real balances of money. Thus,
\[
\begin{align*}
q(z, a) = \begin{cases} q^* & \text{if} \quad q^* \leq \phi m_b \\ \hat{q} & \text{otherwise.} \end{cases}
\end{align*}
\]

As it is shown in the Appendix, for $\gamma = \beta$ agents would carry $m = m^*$, nobody would need to take loans, $a = 0$, and $q(z, 0) = q^*$. On the other hand, if $\gamma > \beta$, then $m < m^*$, and people want to borrow in order to get closer to $q^*$. This means that $a > 0$, and $q(z, a) = \hat{q} < q^*$. Notice that $q^*$ in principle does not depend on $F(\delta)$, since these are idiosyncratic shocks that do not affect the equation that defines $q^*$, namely, $u'(q^*) = c'(q^*) = 1$.

The first order condition for this problem can be expressed as follows,
\[
\phi [u'(q) - 1] = \lambda_q
\] (10)
where $\lambda_q$ is the multiplier assigned to the constraint in problem (7). It is easy to show that in equilibrium, $d = m_b$ and $q = \phi m_b$ always for any $i > 0$.\(^{14}\) This implies that $u'(q) > 1$, the buyer spends all his money, and trades are not efficient.\(^{15}\) It also means that since the budget constraint binds for any $i > 0$, then the interest rate basically acts as a tax on consumption. Finally, using the envelope theorem and equations (6) and (10) we get

$$V_{2j}^I = -\phi (1 + i), \quad V_{2j}^a = \delta, \quad j = b, s.$$  
$$V_{2m}^m = V_{3m}^m = \phi$$  
$$V_{2b}^m = V_{3m}^m + \lambda_q = \phi u'(q). \quad (11)$$

3.3 First Market

Uncertainty is solved right before entering the first market. Net lenders deposit their money with the expectation to earn a nominal interest, $i$. Net borrowers can finance higher consumption at the same rate. Financial intermediation is done by perfectly competitive banks that accept deposits from producers and extend loan contracts to consumers. Agents are anonymous and there is lack of enforcement and certain lack of record keeping.\(^{16}\) In this world, agents do not know anyone else’s trading histories and banks impose that agents can only borrow up to a limit based on the value of collateral. Banks are the only ones who can verify the assets and seize collateral to ensure repayment. In case of default, banks cash the value of collateral into $\delta a / \phi$ units of money, that are automatically transferred back to lenders.

The probability of being a buyer and matching in the second market is $n$. The probability of being a seller and matching as well is $s$. Finally, the probability of being either a seller or a buyer but not matching would then be $1 - n - s$. Thus, an agent who enters the first market with $m$ units of money and $a$ units of the real asset has expected lifetime utility

$$V_1(m, a) = \int [n V_{2b}(m_b, l_b, a) + s V_{2s}(m_s, l_s, a) + (1 - n - s) V_{2o}(m_o, l_o, a)] f(\delta) d\delta \quad (12)$$

\(^{14}\) The optimal case, that analyzes the first-best allocation, is straightforward and has already been discussed.

\(^{15}\) See the Appendix.

\(^{16}\) This also implies that banks cannot punish agents who default on their loan contracts. However, they can impose borrowing limits based on collateral, as it is mentioned next.
where \( m_j = m + l_j, \ j = b, s, o \). Once trading types have realized, the problem for sellers and for buyers who do not match (lenders) is the same, and different from that of buyers who match (borrowers).

**Lenders, types** \( j = s, o \). These types solve the following problem

\[
\max_{l_j} V_{2j}(m_j, l_j, a) \quad \text{s.t.} \quad 0 \leq m + l_j. \tag{13}
\]

The first order condition for this problem is

\[
V^{m}_{2j} + \lambda_j + V^{l}_{2j} = 0
\]

where \( \lambda_j \) is the multiplier corresponding to the constraint. It is not hard to see that these agents will become net lenders in equilibrium, that is \( \lambda_j > 0, \ j = s, o \). Since they will earn a positive interest, it is always profitable for them to lend the money that they will not need.\(^{17}\) Using (11) the previous first order condition becomes

\[
\lambda_j = \phi i, \ j = s, o. \tag{14}
\]

**Borrowers, type** \( j = b \). Buyers face a borrowing constraint that depends on the value of collateral at any point in time. This constraint captures the idea that in an environment with limited enforcement a borrowing limit is established by financial intermediaries: the real amount of loan plus interest must not be greater than the real value of available collateral, \( \phi (1 + i) l \leq \delta a \). The difference here is that each agent will face a different constraint according to the particular value of her collateral. The problem that each buyer of type \( b \) solves is

\[
\max_{l_b} V_{2b}(m_b, l_b, a) \quad \text{s.t.} \quad \begin{align*}
0 & \leq m + l_b \\
l_b & \leq \frac{\delta a}{\phi (1 + i)}
\end{align*} \tag{15, 16}
\]

where the second constraint is the borrowing constraint, with multiplier \( \lambda_{lb} \). The first constraint has multiplier \( \lambda_b \). Again, one can easily see that buyers of

\(^{17}\) As we have already discussed, at \( i = 0 \) their optimal strategy is to deposit any amount in the interval \([0, m]\). However, at \( i = 0 \) people carry the optimal \( m^* \) and there is no credit market.
type \( b \) will always become net borrowers in equilibrium, i.e. \( \lambda_b = 0 \), since money is required to buy consumption.\(^{18}\) Assumptions on preferences, like the usual Inada conditions, guarantee that \( m + l_b > 0 \). Therefore, the first order condition is

\[
V_{2b}' - \lambda_{lb} + V_{2b}m + \lambda_b = 0
\]

which, using (11) and \( \lambda_b = 0 \), can be written as follows

\[
\phi [u'(q) - (1 + i)] = \lambda_{lb} .
\]

Most models of borrowing with collateral consider borrowing constraints that are either always binding or not.\(^{19}\) However, I analyze here two types within the group of buyers of type \( b \) (consumers): those who are borrowing constrained and those who are not. Thus, the constraint will be binding for the first (\( \lambda_{lb} > 0 \)), but not so for the second (\( \lambda_{lb} = 0 \)). In other words, the first order condition for the constrained agents reduces to

\[
u'(q_c) > 1 + i ,
\]

whereas for the unconstrained

\[
u'(q_u) = 1 + i
\]

where \( q_c \) and \( q_u \) are the amounts of good in the decentralized market consumed by constrained and unconstrained consumers, respectively. Equation (19) tells us that unconstrained buyers take loans until the marginal benefit of doing so is equal to its marginal cost. Trades are efficient in this case. Constrained agents are not able to reach this efficiency and will therefore borrow as much as they are allowed to. Therefore, in equilibrium the marginal utility of constrained borrowers will be higher than that of unconstrained agents.

Notice, however, that the fact that for unconstrained borrowers equation (19) holds with equality may wrongly lead to the conclusion that, since their loan

\(^{18}\)Berentsen et al. (2007) prove that financial intermediation improves welfare (consumption) away from the Friedman rule. Not only \( m > 0 \) because \( m > 0 \), but also \( l_b > 0 \).

\(^{19}\)Monacelli (2006) makes technical assumptions to ensure that around the steady state both the budget and borrowing constraints of all the agents are satisfied with equality. As he argues in his paper, this was needed in order to be able to log-linearize the model. Iacoviello (2005) also builds a model such that borrowing constraints could be non-binding only in some uncertainty cases. However, he assumes that uncertainty is sufficiently small so that they will always be binding for all agents.
contracts are efficient, they actually get the efficient consumption \( q_u(z, a) = q^* \). On the contrary, for positive interest rates the equation \( u'(q_u) = 1 + i \) actually means that \( u'(q_u) > 1 \). Therefore, \( q_u(z, a) = \hat{q}_u < q^* \). Nonetheless, as will be shown below, since \( q = \phi(m + l_b) \) and \( u' \) is decreasing in \( q \), consumption for constrained borrowers is such that \( \hat{q}_c < \hat{q}_u < q^* \).

From equation (19), since \( u' \) is continuous and strictly decreasing, and \( \delta \) is continuously distributed, the next lemma follows that separates constrained from unconstrained borrowers.

**Lemma 1.** There exists a critical value, \( \delta_c \), that lies in the interior of the support of the distribution \( F(\delta) \), and is defined by

\[
u'(z + \frac{\delta_c a}{1+i}) = 1 + i.
\]  

Thus, we can obtain the nominal loan demand of both constrained and unconstrained agents, \( l^c_b \) and \( l^u_b \) respectively:

\[
l_b = \begin{cases} 
l^c_b = \frac{\delta a}{\phi(1+i)}, & \text{if } \delta < \delta_c \\
l^u_b = \bar{l}_b, & \text{if } \delta \geq \delta_c 
\end{cases}
\]

where \( \bar{l}_b = \frac{\delta a}{\phi(1+i)} \). Thus, for \( \delta < \delta_c \), \( l_b \) is increasing in \( \delta \), and for \( \delta \geq \delta_c \), \( l_b = \bar{l}_b \), which is constant given that unconstrained borrowers take loans only until the marginal benefit of doing so is equal to its marginal cost. Also, both loan demands are naturally decreasing in the interest rate and \( l^c_b < l^u_b \). Therefore, since \( q \) is strictly increasing in \( l_b \), \( q = z + \phi l_b \), then \( q_c = z + \phi l^c_b < q_u = z + \phi l^u_b \).

On the other hand, market clearing in the credit market requires that the total amount of loans demanded by constrained and unconstrained borrowers cannot exceed the total amount of money deposited by lenders, \( s z + (1-n-s)z = (1-n)z \).

\[
\int_0^{\delta_c} \frac{\delta a}{1+i} f(\delta)d\delta + \int_{\delta_c}^{\delta} \frac{\delta_c a}{1+i} f(\delta)d\delta = \frac{1-n}{n} z.
\]  

\[\text{20 It can be easily derived from equation } u'(z + \phi l^u_b) = 1 + i.\]
3.4 Asset Pricing and Liquidity Premium

I will now turn to the equilibrium equations. Differentiating (12) and using the envelope theorem and equations (14) and (17), the marginal values of money and collateral can be expressed as

\[ V_m^1 = \int \phi \left[ n u'(q) + (1 - n)(1 + i) \right] f(\delta) d\delta , \] (22)

and

\[ V_a^1 = \int [ n \delta \left( \frac{u'(q)}{1 + i} - 1 \right) + \delta ] f(\delta) d\delta . \] (23)

The marginal value of money has the following interpretation. If the agent is a buyer who matches in the second market, an additional unit of money yields \( u'(q) \) units of marginal utility in that market. In any other case the agent can deposit the additional unit of money which will yield the nominal return \((1 + i)\). Notice that the credit market increases the marginal value of money because agents can earn interest on idle cash.

Remember that equation (5) is the pricing equation for the asset that is used as collateral. Now that we have derived the expression for \( V_a^1 \), we know what the asset pricing equation looks like. The price of this asset is always that of the good in the centralized market, 1. However, in accordance with such price, the pricing equation will also reflect both the specific expected return and the liquidity properties of the collateralizable asset: every additional unit of collateral allows us to take loans that will give us net additional utility \( u'(q) - (1 + i) \). Using (5) and the first order condition (23), and rearranging them, we can derive the expression for the liquidity premium of collateral

\[ 1 - \beta \bar{\delta} = \int n \left( \beta \delta \frac{u'(q)}{1 + i} - \beta \delta \right) f(\delta) d\delta . \] (24)

where \( \bar{\delta} \) is the mean value of the distribution. The cost of storing an additional unit of collateral is 1, and the expected return of doing so is \( \beta \int_0^{\bar{\delta}} \delta f(\delta) d\delta = \beta \bar{\delta} \). Thus, the right-hand side of (24) is the collateral’s liquidity premium. However, we imposed that \( \beta \bar{\delta} < 1 \), which means that this premium is positive. Therefore, agents need an incentive to hold collateral other than its investment return. The liquidity properties of collateral provide this incentive.

\[ ^{21} \text{It is straightforward to check that the value function is concave in both } m \text{ and } a. \]
Plugging (22) and (23) into (4) and (5) respectively, and using equation (20) along with the fact that \( u'(q) = 1 + i \) for unconstrained borrowers, we can rearrange to obtain the equilibrium conditions:

\[
\frac{\gamma - \beta(1 + i)}{\beta(1 + i)} = \int_0^{\delta(z,a)} n \left( \frac{u'(z + \frac{\delta a}{1+i})}{1+i} - 1 \right) f(\delta) d\delta, \tag{25}
\]

\[
\frac{1 - \beta \bar{\delta}}{\beta} \geq \int_0^{\delta(z,a)} \delta n \left( \frac{u'(z + \frac{\delta a}{1+i})}{1+i} - 1 \right) f(\delta) d\delta. \tag{26}
\]

The terms in parenthesis on the right-hand sides of equations (25) and (26) reflect the excess of marginal utility with respect to the marginal cost of borrowing in the credit market. Thus, the right-hand side of equation (25) can be interpreted as the average excess of marginal utility over the marginal cost of borrowing among constrained borrowers.

**Definition 1.** Given a distribution function \( F(\delta) \) and an interest rate \( i \), a symmetric stationary monetary equilibrium is a choice of real balances, \( z \), and collateral holdings, \( a \), that satisfy (21) and (26), and a growth rate of money, \( \gamma \), that is consistent with the monetary authority’s policy \( i \) and satisfies (25).

**Proposition 1.** There exists a unique symmetric stationary monetary equilibrium with credit. Equilibrium consumption for both types is decreasing in \( i \) and \( \gamma \), and increasing in the value of collateral, \( F(\delta) \). Finally, in equilibrium consumption by constrained agents is less than that of unconstrained ones, \( \hat{q}_c < \hat{q}_u < q^* \), with \( (\hat{q}_c, \hat{q}_u) \to q^* \) as \( \gamma \to \beta \).

The proof of everything that is stated in this proposition has already been provided in previous sections, except that of existence and uniqueness, which I relegate to the Appendix.

### 3.5 Monetary Policy

I consider in this section a central authority that wants to maximize the welfare of the representative agent, (2). In order to achieve its objective, the central bank must choose \( (x, q_c, q_u, a, \tau) \), and its choice must be consistent with the equations describing the equilibrium, (25), (26), (21), and \( q = \phi(m+l) = z + z_l \). Now, it is straightforward that \( x = x^* \) such that \( U'(x^*) = 1 \). Consumption
by unconstrained borrowers, \( q_u \), is pinned down by the interest rate in (19), and \( \tau \) is uniquely determined by (25) so that it is consistent with the interest rate. Given the lump-sum transfer, \( \tau \), the policy is implemented by choosing the interest rate. Henceforth I will use \( i \) for \( 1 + i \). Notice that, since \( \delta \) is an idiosyncratic shock, the choice variable for the central bank, \( q_c \), does not depend on \( \delta \). Although it may depend, as will be shown, on \( F(\delta) \). In this fashion, let us construct the problem of the central by reducing the equilibrium equations as follows. Market clearing condition can be written as

\[
z = \frac{n}{1 - n} \left[ \frac{a\delta_c^2}{2\delta_i} + \frac{a\delta_c(\hat{\delta} - \delta_c)}{i\delta} \right].
\]  

(27)

On the other hand, if we use (27) in \( q_c = z + z_t \), we have

\[
q_c = z + \int_{0}^{\delta_c} \frac{\delta a}{\delta_i} = \frac{\delta_c a}{\delta_i} \left[ 2n\hat{\delta} + \delta_c(1 - 2n) \right].
\]  

(28)

I show in the Appendix how this expression for \( q_c \) can be used to reduce equations (26) and (25) to the following:

\[
q_c = aJ(i, \tau)
\]  

(29)

where

\[
J(i, \tau) = \frac{\beta\delta_c[n4\hat{\delta} + \delta_c(1 - 2n)]}{2(1 - n)[2i - (\gamma - \beta i)\delta_c]}.
\]  

(30)

Although I will later on discuss policies that involve strictly positive interest rates in more detail, let us now define

\[
i_\theta = \frac{(1 + \tau)\delta_c}{2(1 - \beta \delta) + \beta \delta_c}.
\]  

(31)

As will be shown below, in the range \( i \geq i_\theta \) people decide not to store any amount of collateral, \( a = 0 \), and money grows at a higher rate than that of time preference, \( \gamma > \beta \). In this case, \( q_c \) would be determined by satisfying (25),\(^{22}\)

\[
q_c^m(\tau) = u'\left( \frac{1 + \tau}{\beta n\delta_c} \right).
\]  

(32)

---

\(^{22}\) Note that at any period of time the central authority can take \( m \) and \( a \) as given, optimally chosen in the centralized market in the previous period. Thus, \( \delta_c \) would be determined only by \( i \) through equation (20).
On the other hand, the highest possible $q_c$ would be achieved at $i = 1$, and it would be defined by

$$q_c^F = u^{-1}(1).$$

(33)

Therefore, the central authority is restricted to implement policies such that $q_m^c(\tau) \leq q \leq q_c^F$. Finally, the problem to be solved by the authority responsible for economic policy could be written as follows:

$$\max_{q_c, a, \tau} \quad n \left[ \int_0^{\delta_c} u(q_c) + \int_{\delta_c}^{\hat{\delta}} u(q(i)) \right] - s \left[ \int_0^{\delta_c} q_c + \int_{\delta_c}^{\hat{\delta}} q(i) \right] + (\beta\hat{\delta} - 1)a$$

subject to

$$q_c = aJ(i, \tau)$$

and

$$q_m^c(\tau) \leq q \leq q_c^F.$$ 

(34)

Notice that solving this problem involves the computation of definite integrals. In order to do so, we need to assume a distribution function for the value of collateral. To obtain as general results as possible, and since most related papers employ the same distribution, I assume that agents are uniformly distributed between 0 and $\hat{\delta}$. The following proposition describes monetary policy from our central bank’s point of view.

**Proposition 2.** If $1 + \tau = \beta$, then $i = 1$ is the optimal policy. In this case, $z = z^*$, $a = 0$, and $q_j$ is determined by equation $u'(q_j) = 1$, $j = c, u$. If $1 + \tau > \beta$, then $i > 1$, money balances are such that $0 < z < z^*$, and the optimal policy is $i \in (1, i_0)$.

- $i \geq i_0$ if and only if $a = 0$, with $m > 0$. In this case no borrowing happens, $q_c$ is defined in (32), and it is never optimal to run this policy.
- $1 < i < i_0$ if and only if $m, a > 0$. In this range, $q_c$ is such that $q_c < q_c^F$, and solves the first order condition for the central authority’s problem.
- For very low expected return to collateral, $\overline{\delta}$, the best policy is always $i = 1$. However, away from the Friedman rule the best policy, $i \in (1, i_0)$, is increasing in the expected return to collateral.

This proposition recalls that in this model, given the bargaining protocol, $1 + \tau = \beta$ implements the efficient allocation with agents carrying the efficient

---

23 I denote it by $F$ because it would be the consumption implemented under the Friedman rule.
amount of money, $m = m^*$, so that $u'(q^*) = 1$, where $q^* = z^*$. Nobody stores collateral, $a = 0$. Besides, since the term related to $a$ in the welfare function is negative, when the credit market is closed, $i = 0$, it is best to set $a = 0$.

Beyond that, the proposition explains that if interest rates that is too high, $i > i_\emptyset$, even though lenders would obtain higher earnings, it is no longer socially profitable to store collateral, $a = 0$. The interest payments would just be too heavy a load for borrowers. High interest rates would then discourage the storage of the real asset. Also, since higher interest rates reduce consumption, beyond a certain level, $i_\emptyset$, the marginal utility derived from the collateralization of the asset would not be enough to face interest payments. Only for reasonable positive interest rates, $1 < i < i_\emptyset$, would agents be willing to store a positive amount of the real asset.

Furthermore, if the feasible range for polices was such that $1 + \tau > \beta$, some results are particularly relevant. As it is shown in the Appendix, the optimal interest rate is increasing in the value of collateral. This can also be seen by looking at equation (31). The critical $i_\emptyset$ is increasing in $\delta$ and $\tau$. Thus, the higher the value of collateral, the larger the set of monetary policies, $(1, i_\emptyset)$. According to this result, the central bank should pursue a stabilizing policy: if the value of the assets being used as collateral increased, that should be related to higher interest rates. On the other hand, if the value of collateral fell, we should see lower interest rates. This should loosen borrowing constraints and facilitate higher consumption.

Notice that the case of a drop in the value of real assets would be similar to what we are now witnessing in the US and most European housing markets. The Fed has actually been sort of following this policy, so far. They are also encouraged to do so in order to get out of a crisis. This was not the case, however, a few years ago when the value of houses were rising and the Fed cut interest rates. We now know that such policy led to an issue of mortgages, and other derived financial products, probably larger than it would have been desirable. The model that I present here would have advised the Fed to increase interest rates instead in order to prevent excessive borrowing.\footnote{Because in practice people acquire houses not only for consumption but also for investment purposes, establishing this comparison between housing and the real asset in my model may not be completely accurate. However, I believe that the example still serves its illustrative purpose.} A tightening policy might have retarded a necessary recovery at that time, but in accordance to this result it might also have avoided the spiral of mortgage issuance that led to a far more important crisis. A similar way of interpreting this result is that the
central bank should probably pay attention to movements in the value of the assets that are used as collateral when designing monetary policy. This is to a certain extent in contrast with the conclusion in Bernanke and Gertler (1999, 2001) that monetary policy should not respond to changes in asset prices, unless they help to forecast inflationary pressures.

In a similar fashion I show in the Appendix that the critical $\delta_c$ decreases with the value of collateral. In other words, as the asset used as collateral gains value, there are less people borrowing constrained. Finally, I also obtain relevant implications that heterogeneity bears for monetary policy. As $\delta_c$ falls, the interest rate increases. That is, in an economy with a growing population of unconstrained borrowers we would be more likely to see increasing interest rates. Inversely, in an economy with a majority of constrained borrowers interest rates would tend to be lower. This relates to a previous result. If monetary policy is to be accommodative, as suggested by $\partial i/\partial \delta_c > 0$, interest rates would also respond to changing patterns in the borrowing capacity of the population. This digs out a very interesting insight. To be more precise, suppose that the value of collateral fell. The fraction of constrained borrowers, $\delta_c/\hat{\delta}$, would rise. This would increase the liquidity premium of collateral. And we know that this should correspond to lower interest rates to relax the borrowing constraints of an increasing fraction of population. In other words, the liquidity premium, as expressed by (26), does not only depend on the difference between the price of the asset, in terms of the centralized consumption good, and its expected return. This is what standard asset pricing theory would tell us. It also depends on the degree of borrowing heterogeneity, reflected by $\delta_c$. Therefore, according to our asset pricing theory, the liquidity premium of collateral reflects the indirect liquidity that consumers obtain in the decentralized market. Most models are not able to deliver this result. But even more interesting is how I obtain that this premium also depends on the degree of borrowers heterogeneity.

This is another big difference with Berentsen and Monnet (2009), and where one of the main policy implications in my model arises. First off, Berentsen and Monnet obtain a different set of equilibrium allocations. They analyze optimal monetary policy in a channel system, so there are many things different from my model. In their paper even if agents decide not to carry any money, $m = 0$, they can still obtain cash borrowing from the central bank by collateralizing the real asset. I do not contemplate such a facility so there cannot be equilibria in my model with $m = 0$. Consequently, they obtain a different scheme for monetary policy. More noticeably, they obtain that a decrease in the return
to collateral rises the liquidity premium of the asset, and this relates to higher relative money market rates.

We have to remember that even though collateral increases consumption, and therefore welfare, it is costly for society, \(\beta \delta - 1 < 0\). Thus, Berentsen and Monnet conclude that if the return to collateral is low, it will probably not help to increase consumption much. Thus, the optimal policy is to discourage its use by setting high interest rates. As opposed to this, monetary policy in my model would have interest rates being lower. I believe that this is because in my model a low value of collateral, low \(\hat{\delta}\), also means that the fraction of constrained borrowers will be large, and lower interest rates will help to increase consumption for a majority of population. That is, the overall effect of cutting interest rates in the social welfare would, in this situation, be greater than the cost of collateral effect.

Finally, it is worth mentioning that in papers like Kiyotaki and Moore (1997), or Ferraris and Watanabe (2008), the asset used as collateral is also an input to production (land and capital respectively). Whereas agents in my model only borrow to finance consumption, agents in those other papers borrow also for investment purposes, and this is reflected in the equilibrium equations. While this modeling approach is in some sense simplifying, it has allowed me to focus on the analysis of the pure decision to hold an asset that may be collateralized rather than issues like optimal accumulation of capital and such, for which other frameworks are more suitable. In this paper the asset that serves as collateral and money are complements. That is, the asset is held partially for its return, but mainly for the purpose of acquiring money (loans). It has no other role relating to production or investment that makes it worth deviating resources from money balances and into those other assets in the presence of inflation. Thus, if money loses value people are less willing to hold collateral for the motive of acquiring money.

4 Conclusion

The purpose of this paper was to properly analyze the role of the value of collateral and its interaction with monetary policy in an environment with heterogeneous borrowers. A real asset is introduced that presents different features from those of fiat money and a standard financial asset and still coexists with money due to its value as a collateralizable asset in contingencies. This indirect
liquidity is reflected in the asset pricing equations.

The Friedman rule is the optimal policy. When money grows at this rate agents behave efficiently by storing nothing of the asset. Away from the Friedman rule, agents are willing to hold collateral to increase consumption as long as interest rates are moderate. Monetary policy smooths consumption by accommodating changes in the value of the asset used as collateral. The asset pricing theory in the model maintains that the liquidity premium of collateral depends not only on the indirect liquidity provided by the asset, but also on the degree of borrower’s heterogeneity. A drop in the value of assets would increase the fraction of borrowing constrained agents, and this acts to rise the liquidity premium. These aspects are not present in other models of collateral. On the other hand, higher liquidity premium corresponds to lower interest rates.

This framework may be particularly useful to understand the market of asset-backed securities that has so rapidly developed in the last years, and recently collapsed. In a currently ongoing project that builds on Geromichalos, Licari, and Suarez-Lledo (2007) we develop a model in which a financial asset serves as collateral in a credit market where agents obtain bonds, instead of money, against the value of their collateral. The model should deliver explicit relations between monetary policy and changes in both the price and liquidity of the asset-backed security and the underlying asset. On a different scale, this type of model structure may be suitable for the study of the collateral frameworks of central banks.

A Appendix

Welfare Function. When we discussed the first-best allocation I stated the welfare function that any social planner, in our case the central bank, would have to maximize in order to achieve its objective. I now show how to derive the welfare function, equation (2). Recall that \( M_{s+1} = M + \tau M \), and that the portfolio of a buyer entering the third market is \((m, l, a) = (0, l_b, a)\), and that of a type \( j = s, o \) is \((m, l, a) = (m + \tau M + l_b, l_s, a)\). First, we need hours worked:

\[
\begin{align*}
    h_b &= x - (E[\delta] - 1)a + \phi(1 + i)l_b \\
    h_s &= x - (E[\delta] - 1)a - \phi(1 + i)l_s
\end{align*}
\]

Total hours worked \( h = nh_b + (1 - n)h_s \). Using the market clearing condition, this expression reduces to \( h = x - (E[\delta] - 1)a \). Thus, welfare can be computed
as
\[ W = U(x) - x - a + n \left[ \int_{0}^{\delta} u(q_c) + \int_{\delta}^{\tilde{\delta}} u(q_a) \right] - s \left[ \int_{0}^{\delta} q_c + \int_{\delta}^{\tilde{\delta}} q_a \right] + \\
\sum_{t=1}^{\infty} \beta^t \left\{ n \left[ \int_{0}^{\delta} u(q_c) + \int_{\delta}^{\tilde{\delta}} u(q_a) \right] - s \left[ \int_{0}^{\delta} q_c + \int_{\delta}^{\tilde{\delta}} q_a \right] + U(x) - x + (E[\delta] - 1)a \right\} \\
= \frac{U(x) - x + n \left[ \int_{0}^{\delta} u(q_c) + \int_{\delta}^{\tilde{\delta}} u(q_a) \right] - s \left[ \int_{0}^{\delta} q_c + \int_{\delta}^{\tilde{\delta}} q_a \right] + (\beta E[\delta] - 1)a}{1 - \beta} \\
\]

\textbf{In equilibrium} \( q = \phi m_b \) \textbf{for any} \( i > 0 \). Suppose there exists an equilibrium with \( i > 0 \) and \( q < \phi m_b \), where \( m_b = m + l_b \). At any equilibrium like this the amount consumed \( q \) that solves the problem in market 2 is optimal. That is, \( q = q^* \) and is defined by \( u'(q^*) = c'(q^*) = 1 \). Remember that the problem to be solved in the centralized market is

\[
\max_{x,h,m_{+1},a_{+1}} U(x) - h + \beta V_1(m_{+1}, a_{+1}) \\
\text{s.t.} \quad x + \phi m_{+1} + a_{+1} = h + \phi m - \phi(1+i)l + \delta a.
\]

Also, the total utility in a certain period of time is given by

\[
U(x, h, q) = U(x) - h + nu(q) - (1 - n)q.
\]

Thus, since \( q = q^* < \phi m + \phi l_b \), and given the assumptions on the utility functions, we can always consider a stationary equilibrium in which someone decided to reduce her debt by an amount \( dl \), small enough such that she is still in the neighborhood of the optimal \( q \), therefore still able to consume \( q^* \). The budget constraint would still be non-binding and the variation in total utility would be

\[
dU = \phi(-dl) - \phi(1+i)(-dl) = \phi(1+i)dl - \phi dl = i \delta l \geq 0.
\]

Then, \( dU > 0 \) for any \( i > 0 \). Therefore we could not have any equilibrium with \( i > 0 \) and \( q < \phi m_b \), and the buyer’s budget constraint is always binding in equilibrium.

\[ \square \]
Proof of Lemma 1. Since \( z \) and \( a \) are chosen in the previous period, and \( \delta \) is realized in the present period, we can take \( z \) and \( a \) as given and define the following function,

\[
f(\delta) = u'\left(z + \frac{\delta a}{1+i}\right) - 1 + i,
\]

where the term \( 1 + i \) is constant. All we have to prove is that there is \( \delta^* \) such that \( f(\delta^*) = 0 \). Given the maintained assumptions of the model we know that \( u'(0) = +\infty, u'(+\infty) = 0 \), and that \( u' \) is strictly decreasing in \( q \), since \( u'' < 0 \). Thus, \( f \) is also continuous and strictly decreasing in \( \delta \).

For any given \((z,a)\), as \( \delta \) becomes large, \( u'(\delta) \to 0 \). For \( \delta \) big enough, \( u' = 0 \), which implies that \( f(\delta) < 0 \). On the other hand, as \( \delta \to \delta_0 \) (where \( \delta_0 \) is small enough, but remains positive) it can only be the case that \( u'(z) \geq 1 + i \), since \( z \) and \( a \) are chosen optimally. Thus, if \( z \) and \( a \) were such that \( u'(z + \delta_0 a/(1+i)) = 1 + i \), then we are done. Finally, if they were such that \( u'(z+\delta_0 a/(1+i)) > 1+i \), then by the Intermediate Value Theorem we have that for any term \((z,a,i)\) there exists a unique \( \delta^* \) such that \( f(\delta^*) = 0 \), and \( \delta^* = \delta_c \).

For no matter how small the interest rate is, we can always find a sufficiently large \( \hat{\delta} \) such that for some agent \( f(\hat{\delta}) < 0 \). Therefore, there will always be a positive mass of unconstrained agents. \( \Box \)

Proof of Proposition 1. Since \( \gamma \) is uniquely pinned down by (25), the equilibrium boils down to two equations in two unknowns: (26) and (21) in \( z \) and \( a \). Thus, in order to prove existence and uniqueness of an equilibrium basically three things need to be shown: that these two curves are strictly monotonic,
that they have different slopes, and finally that they actually intersect in the plane \((z, a)\).

First, I want to obtain the slope \(dz/da\). It turns out that while total differentiation is enough to draw conclusions from (21), the implicit function theorem needs to be applied to equation (26). If we express equation (26) as

\[
\rho(z, a) = \int_0^{\delta_c(z, a)} \delta n \left( \frac{u'(z + \frac{\delta a}{1+i})}{1 + i} - 1 \right) dF(\delta) - \frac{1 - \beta \bar{\delta}}{\beta} = 0
\]

then, according to this theorem, such slope would be

\[
\frac{dz}{da} = -\frac{\partial \rho(z, a)}{\partial a} = -\frac{\int_0^{\delta_c} \delta n \left( \frac{u''(\frac{\delta a}{1+i})}{1+i} \right) dF(\delta)}{\int_0^{\delta_c} \delta n \left( \frac{u''(\frac{\delta a}{1+i})}{1+i} \right) dF(\delta)} < 0. \tag{a.1}
\]

We know that from the assumption that \(u'' < 0\). Then, both numerator and denominator are strictly negative. This implies that the slope \(dz/da\) is strictly negative. Therefore, I have shown that equation (26) has negative slope and is strictly monotonic.

Let us now turn to equation (21). Applying total differentiation to this expression and by the Leibniz’s rule we obtain

\[
\frac{dz}{da} = \frac{n}{1 - n} \left[ \int_0^{\delta_c(z, a)} \frac{\delta}{1 + i} dF(\delta) + \int_{\delta_c(z, a)}^{\hat{\delta}} \frac{\delta_c}{1 + i} dF(\delta) \right] > 0. \tag{a.2}
\]

Therefore, it has been shown that equation (21) has positive slope and is strictly monotonic. Thus, these two curves have slopes with opposite sign and, as it is obvious from inspection of equations (a.1) and (a.2), the magnitude of the slopes is clearly different. It only remains to show that they actually intersect.

In the model the only feasible choices of real balances and asset holdings are those with positive values. However, we have to make sure that this is indeed the case and these two curves intersect in the first quadrant of the plane \((z, a)\). In addition, even though we know that one curve is upward sloping and the other one is downward sloping, we have to rule out cases where, for instance, the upward sloping curve would start above the other one and so they would not intersect. Notice that at \(z = 0\), equation (21) becomes

\[
\left[ \int_0^{\delta_c(z=0,a)} \frac{\delta}{1 + i} dF(\delta) + \int_{\delta_c(z=0,a)}^{\hat{\delta}} \frac{\delta_c}{1 + i} dF(\delta) \right] a = 0,
\]

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and since the term in brackets is strictly positive, when \( z = 0 \) equation (21) tells us that \( a = 0 \). On the other hand, equation (26) can also be analyzed at \( z = 0 \):

\[
\frac{1 - \beta \tilde{\delta}}{\beta} = \int_{0}^{\delta_c(z=0,a)} \delta \ n \left( \frac{u'(\frac{\delta a}{1+i})}{1+i} - 1 \right) \, dF(\delta),
\]

which can be rewritten as

\[
\frac{1 - \beta \tilde{\delta}}{\beta} + \int_{0}^{\delta_c(z=0,a)} \delta \ n \ dF(\delta) = \frac{n}{1+i} \int_{0}^{\delta_c(z=0,a)} \delta \ u' \left( \frac{\delta a}{1+i} \right) \ dF(\delta).
\]

Notice that we integrate over \( \delta \) and \( u' \) is monotonically decreasing. Thus, \( a \) can always be factored out of the integral. Also, both sides of the equation are strictly positive. Therefore, at \( z = 0 \), we can conclude that \( a > 0 \). That is, the choice of \( a \) that satisfies equation (26) is positive. This proves that in the plane \((z, a)\) equation (26) starts from above (21) and strictly decreases, while (21) strictly increases. It turns out quite hard to develop the last expression any further in order to obtain a cleaner solution for \( a \). However, examples of how this last part of the proof would work with a particular functional form for \( u' \) can easily be provided. Therefore, we can finally conclude that the two curves will intersect and a unique equilibrium exists.

\[\square\]

**Derivation of** \( J(i) \). As it was mentioned previously, the choice variable for central bank is \( q_c \), which does not depend on \( \delta \) because it is an idiosyncratic shock. Although it may depend, as will be shown, on \( F(\delta) \). Thus, equations (26) and (25) can be written as follows,

\[
\frac{1 - \beta \tilde{\delta}}{\beta} \geq \frac{\delta_c^2}{2 \tilde{\delta}} \ n \left( \frac{u'(q_c)}{i} - 1 \right) \tag{a.3}
\]

and

\[
\frac{\gamma - \beta i}{\beta i} = \frac{\delta_c}{\delta} \ n \left( \frac{u'(q_c)}{i} - 1 \right). \tag{a.4}
\]

If we plug \( u'(q_c) \) from (a.3) in (a.4), and rearrange, we obtain

\[
\gamma - \beta i = 2i \left( \frac{1}{\delta_c} - \frac{\beta \tilde{\delta}}{2 \delta_c} \right)
\]

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Now we substitute $\tilde{\delta}/c$ from (28) in the previous equation to arrive at

$$q_c = a \frac{\beta \delta_c [n4\tilde{\delta} + \delta_c(1-2n)]}{2(1-n)[2i - (\gamma - \beta i)\delta_c]}$$

---

**Proof of Proposition 2.** Due to the similar structure of the problem, the style of the proof of some of the statements in this proposition resembles to some extent that in Berentsen and Monnet (2009). I begin by addressing the first paragraph.

If $1 + \tau = \beta$, $i = 1$ is the optimal policy. First off, $1 + \tau = \beta$ implies $i = 1$. Since $i \geq 0$, if $\gamma = \beta$ implied $i > 1$, then $\beta(1+i) > \beta$, and therefore $\gamma - \beta(1+i) < 0$. However this makes the LHS of equation (25) negative, which implies, since this equation holds with equality, that $u'(q_c) < 1 + i$, which is a contradiction.

Assume now that $1 + \tau = \beta$ and $i = 1$, but suppose that this policy is not optimal, i.e., suppose that $1 + \tau = \beta$ and $i = 1$ implies that $u'(q_c) > 1$. If this was the case, then (25) would be written

$$\frac{1 + \tau - \beta}{\beta} = \int_0^{\delta_c} n [u'(q_c) - 1] f(\delta) d\delta$$

and its RHS would be strictly positive. Therefore, the LHS would also be strictly positive, implying $1 + \tau > \beta$, which is a contradiction. Thus, for $\gamma = \beta$, $i = 1$ is the optimal policy, and the money growth rate consistent with such policy is $\tau = \beta - 1$.

If $1 + \tau > \beta$, then $i > 1$. Suppose on the contrary that $1 + \tau > \beta$ implied $i = 1$. Then, equation (25) implies $u'(q_c) > 1$, which is fine since $\gamma > \beta$ and the optimal allocation can only be attained at $\gamma = \beta$. However, we know that in equilibrium $u'(q_u) = i$, which in this case reads $u'(q_u) = 1$. However this contradicts $u'(q_u) > 1$ implied by $\gamma > \beta$.

Notice that for $i = 1$, consumption $q_c$ does not depend on $i$. Suppose now that we wanted to not only derive the optimal policy but also to analyze how monetary policy depends on the return to collateral, and what would be the best policy away from the Friedman rule. For this range of monetary policies,
$q_c$ must satisfy (a.4) and $q_c = aJ(i)$. Therefore, if we use $i$ from (a.4) and $a$ from $q_c = aJ(i)$, the central bank’s problem can be expressed as

$$
\max_{q_c} n \left[ \int_0^{\delta_c} u(q_c) + \tilde{\delta} q_c \right] - s \left[ \int_0^{\delta_c} q_c + \tilde{\delta} q_u \right] + (\beta - 1) \frac{q_c}{J \left( \frac{u'(q_c) \hat{n} \delta_c - \gamma \hat{\delta}}{n \delta_c - 2 \hat{\delta}} \right)}
$$

and

$$
q_c^m(\tau) \leq q \leq q_c^F, \quad \text{(a.5)}
$$

I use the Lagrangian method with $\mu_m$ and $\mu_F$ being the lagrange multipliers of the first and second constraint, respectively. From the proof of proposition 1 and by the Kuhn-Tucker sufficiency theorems, we know that there exists a unique solution to this problem and that it is given by the first-order condition (FOC) of the Lagrangian problem. When the central bank solves for $q_c$ obtains the following FOC:

$$
\left[ nu'(q_c) - s \right] \delta_c / \hat{\delta} + \frac{(1 - \beta \hat{\delta})}{J(i)} \left[ J'(i) \left( i \cdot \frac{u'(q_c)}{u'(q_c)} + \frac{u'(q_c)}{u'(q_c)} \beta(n \delta_c) \right) q_c - 1 \right] = \mu_F - \mu_m
$$

Now, suppose that the best policy was to implement $q_c^m(\tau)$, that is, $i \geq i_\emptyset$. In this case only (a.4) holds with equality. If we substitute $u'(q_c)$ from (a.3) in (a.4) and rearrange, we obtain

$$
i \geq \frac{(1 + \tau) \delta_c}{2(1 - \beta \hat{\delta}) + \beta \delta_c} = i_\emptyset.
$$

On the other side of the implication, if we assumed that $i \geq i_\emptyset$, and we had both $m > 0$ and $a > 0$, this would mean that (a.3) and (a.4) held with equality. If we play around with them we would arrive at

$$
i = \frac{(1 + \tau) \delta_c}{2(1 - \beta \hat{\delta}) + \beta \delta_c},
$$

which would contradict $i \geq i_\emptyset$. Let us refer to the left-hand side of the FOC as $F(q_c^m(\tau), \hat{\delta})$. If such was the best policy, that would imply $\mu_F = 0$, $\mu_m > 0$, and therefore that $F(q_c^m(\tau), \hat{\delta}) < 0$. Let us check if that is true.

$$
\lim_{i \to i_\emptyset} J(i) = \frac{(n4\hat{\delta} + \delta_c(1 - 2n)) \left(2(1 - \beta \hat{\delta}) + \beta \delta_c\right)}{4(1 - n)\gamma \hat{\delta}} > 0.
$$

Also,

$$
\lim_{i \to i_\emptyset} J'(i) = -\frac{\beta \delta_c \left(n4\hat{\delta} + \delta_c(1 - 2n)\right) \left(2 + \beta \delta_c\right)}{2(1 - n) \left(-\gamma \delta_c + (2 + \beta \delta_c)i_\emptyset\right)^2} < 0.
$$
On the other hand, we also have
\[ \lim_{i \to \emptyset} |J'(i) \cdot i| = J(i_0) \cdot \frac{(2 + \beta \delta_c)i_0}{-\gamma \delta_c + (2 + \beta \delta_c)i_0} > J(i_0). \]

Now, since we know that \( u''(q)/u'(q) = -1 \) for all utility functions exhibiting Decreasing Absolute Risk Aversion (logarithmic and many others), then the whole expression between brackets in the FOC would have a positive sign.\(^{26}\) However, this means that \( F(q_c^m(\tau), \tilde{\delta}) > 0 \), which is a contradiction. Therefore we conclude that, away from the Friedman rule, the best monetary policy must be such that \( i < i_0 \). In other words, \( q_c > q_c^m(\tau) \). Furthermore, whenever monetary policy is such that \( 1 < i < i_0 \), agents hold both money and collateral, \( m, a > 0 \). The inverse is also true. The proof for this follows in very much the same fashion as the one for the previous statement.

Now, let us go back to the policy, \( i = 1 \). In this case
\[ \lim_{i \to 1} J'(i) < 0; \quad \lim_{i \to 1} J'(i) \left( u''(q_c^F) + u''(q_c^F) \frac{\tilde{\delta}}{n\delta_c - \delta} \right) > J(1). \]

This implies that \( F(q_c^F, \tilde{\delta}) > 0 \), which is consistent with \( \mu_F > 0 \) and \( \mu_m = 0 \). Under this policy,
\[ \lim_{\tilde{\delta} \to 0} F(q_c^F, \tilde{\delta}) = \frac{2(1-n)(2 + \beta - \gamma)}{\beta(1-2n)} \left( -\frac{2 + \beta - \gamma}{2 + \beta - \gamma} u''(q_c^F)q_c^F - 1 \right) > 0. \]

This implies\(^{27}\) that for very low values of collateral, \( \mu_F > 0, \mu_m = 0 \), and monetary policy implements \( q_c = q_c^F \), with \( i = 1 \). However, for higher values of the asset used as collateral
\[ \lim_{\tilde{\delta} \to 1/\beta} F(q_c^F, \tilde{\delta}) = 0, \]
and this implies \( \mu_F = 0 \). That is, \( q_c^m(\tau) < q_c < q_c^F \), with \( i \in (1, i_0) \), where \( q_c \) solves \( F(q_c, \tilde{\delta}) = 0 \).

\(^{26}\) Notice also that
\[ \frac{u''(q_c)}{u'(q_c)} \frac{\gamma \tilde{\delta}}{\beta(n\delta_c - \delta)} > 0. \]

\(^{27}\) That \( \tilde{\delta} \to 0 \) basically implies that \( \delta_c \to 1 \). In other words, as collateral loses value, there are more people borrowing constrained.
That the best policy, away from the Friedman rule, is increasing in the expected return to collateral can also be deduced from the following. If we take derivatives,

\[ \frac{\partial i_\varnothing}{\partial \delta} = \frac{2(1 + \tau)\delta_c \beta \frac{\partial \delta}{\partial \delta_c}}{[2(1 - \beta \delta) + \beta \delta_c]^2} > 0 \]

since, obviously, \( \partial \delta / \partial \delta > 0 \). Thus, if \( i_\varnothing \) increases with the value of collateral, the possible range for monetary policy, \((1, i_\varnothing)\), is enlarged. Moreover,

\[ \frac{\partial \delta}{\partial \delta_c} = \frac{1}{2} \left( \frac{\beta i_\varnothing - \gamma}{\beta i_\varnothing} \right) < 0 \]

and therefore \( \partial \delta / \partial \delta_c < 0 \), since they have the same sign. A standard chain rule will then yield \( \partial i_\varnothing / \partial \delta_c < 0 \).

\[ \square \]

References


