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Restricted environments and incentive compatibility in interdependent values models[1]

by

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Abstract: We study the possibility of designing satisfactory ex post incentive compatible single valued direct mechanisms in interdependent values environments, characterized by the set of agents' type profiles and by their induced preference profiles. For environments that we call knit and strict, only constant mechanisms can be ex post (or interim) incentive compatible. For those called partially knit, ex post incentive compatibility extends to groups, and strategy-proofnes implies strong group strategy-proofness in private values environments. The results extend to mechanisms operating on non-strict domains under an additional requirement of respectfulness. We discuss voting, assignment and auctions environments where our theorems apply.

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1 Introduction

A major concern when designing economic mechanisms is to provide agents with incentives to reveal their true characteristics. Setting aside some obviously unsatisfactory solutions, it is well understood that attaining this objective is not always possible. Moreover, when it is, a conflict often arises between the mechanisms' efficiency and incentive compatibility. These generic statements hold for different formulations of the mechanism design problem, and for various concepts of equilibrium.

In this paper we consider situations where agents' values may be interdependent, and their preference profiles only fully determined once the joint profile of types is known. In that context, we study the possibility of designing direct mechanisms that are ex post incentive compatible. Since that property does not require Bayesian updating, we work in a framework where agents' preferences are ordinal.

We start from the remark that a mechanism can only meet attractive lists of desiderata if the class of problems to be dealt with is somewhat constrained. In those cases where types are identified with preferences and agents' values are not interdependent, we can properly refer to these constraints as domain restrictions. In the general case of interdependent values, mechanisms are defined as functions assigning an alternative to each profile of types, but the analysis of their incentive properties requires to know, in addition, the functional relation between type profiles and preference profiles, that we call the preference function. Because of that, we define environment as pairs, formed by the class of type profiles in the domain of the mechanism, and also by the preference function that applies in each case. Our restrictions will be predicated on environments, rather than only on domains.

In addition to this important nuance regarding the objects on which restrictions must be formulated, we would like to emphasize the generality of our approach.

The type of restrictions we impose on environments are quite abstract, because we look for the common features of families of environments, rather than those directly suggested by single applications. The classes of environments that we are about to describe informally, and rigorously define in the next section, are suggested by a careful analysis of a variety of possibility and impossibility results that arise in different fields of application. While the models that are proposed in each case may look quite unrelated at first glance, our approach allows us to go beyond their specific features, and to identify essential and common characteristics.

We define two classes of environments that we call knit and partially knit. Both must meet requirements regarding the possibility to connect admissible pairs of type profiles through sequences of changes in individual types, which are defined in reference to certain alternatives and through the use of the preference function. The set of pairs of type profiles and reference alternatives for which the requirements must be met for an environment to be knit is larger than for it to be partially knit. Thus, the latter is a weaker condition.[1]

[1] The purpose of our introduction is to present the reader with a general roadmap. The details regarding what we exactly mean by the terms connecting pairs of type profiles, or adequate conditions are provided in the formal definitions in Section 2, and clearly illustrated in the analysis of examples of applicatons in Appendix B. Similar caveats apply to other terms that may be used loosely here and will be made precise in the coming sections.

Let's now describe the demands we impose on mechanisms, in order to consider them satisfactory. One first attractive and well-studied requirement is that of ex post incentive compatibility[2], guaranteeing truthful revelation of types to be a Nash equilibrium in all the games that result from any specification of possible type profiles. We also introduce a second concept, that of ex post group incentive compatibility, under which truthful revelation is required to be a strong Nash equilibrium. These are our main target properties, and we can obtain possibility and impossibility results regarding them, for those environments that we call strict, where agents are never indifferent between alternatives. In the general case where some agents may be indifferent among several alternatives, we need to use an additional condition that we call respectfulness. This condition, when applied to private values is a distant relative of non-bossiness (Satterthwaite and Sonnenschein, 1981), but much less demanding than this or other similar conditions analyzed in Thomson (2016). It is mostly required to avoid manipulations by one agent that could benefit others while not gaining anything in exchange.

We present two main types of results, regarding environments that are knit or partially knit, respectively. Two of them are in the vein of impossibility theorems. Theorem 1 states that only constant mechanisms can be ex post incentive compatible and respectful in knit environments. Corollary 1 reaches the same conclusion for strict environments without need to invoke respectfulness, which trivially holds in that case. In fact, the results only apply to the case of interdependent values because, as we prove later on, no environment can be knit in the particular case of private values. The informed reader will observe that the conclusion of our theorem is the same that was obtained by Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006), but the analogy stops here, since the context and the assumptions in each case are very different. Also notice that, since we work with single valued direct mechanisms, our environments are separable in the sense of Bergemann and Morris (2005), and their Corollary 1 applies: no mechanism is interim incentive compatible unless it is ex post incentive compatible. Because of that, Theorem 1 and Corollary 1 have direct implications on the weaker interim notion, with no need to be explicit about agents' beliefs.

Our Theorem 2 can be read as being positive or negative, depending on the specific context of application. Corollary 2 reaches the same conclusion for strict environments without need to invoke respectfulness. It states that all respectful and ex post incentive compatible mechanisms for a partially knit environment will also be ex post group incentive compatible. This result applies both in the case of interdependent and that of private values. They allow us to better understand the possibility of achieving some form of efficiency through the use of incentive compatible mechanisms. To see that, first notice that ex post group incentive compatibility implies Pareto efficiency on the range of the mechanism. This may not be much to say in some cases: for example, if the range is small relative to the whole set of alternatives, or when the mechanism is dictatorial. But we shall exhibit examples of

[2] The study of incentive compatibility in Bayesian terms was started by d'Aspremont and Gérard-Varet (1979), and Arrow (1979), and its appropriate formulation and results depend on the information that will be available to the agents at the time where the analysis is carried out. The case of interdependent values was first studied by D'Aspremont, Crémer, and Gérard-Varet (1990). The notion of ex post incentive compatibility corresponds to the time where agents have received all possible information, and can be defined without attributing cardinal utility to agents, as it does not require Bayesian update. See Jackson (2003).

full range and far from dictatorial Pareto efficient mechanisms that are ex post incentive compatible in partially knit environments.

It is also important to recall that, in the case of private values, ex post incentive compatibility is equivalent to strategy-proofness. Likewise, ex post group incentive compatibility becomes equivalent to strong group strategy-proofness. Hence, a corollary for the case of private values is that, under the conditions of our second theorem, individual and strong group strategy-proofness become equivalent. This parallels results that we obtained in Barberà, Berga, and Moreno (2010, 2016) connecting individual and weak group strategy-proofness.[3]

Our discussion has been abstract till now, but we already said that our results are based in a careful analysis of a variety of problems that arise in different settings, and in specific models that are inspired by essential contributions to several fields of application. We illustrate this by providing examples of situations where our results apply. The examples come in pairs. Two of them refer to deliberative juries and are inspired in our reading of Austen-Smith and Feddersen (2006). Our second pair of examples adress the problem of assigning indivisible objects as in Che, Kim, and Kojima (2015). The last two examples refer to auctions, following the trail of Dasgupta and Maskin (2000) and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006).

We attach much importance to these examples for several reasons. One reason is that they show the unifying power of our approach. The models we get inspiration from look very different from each other, because they describe the types of agents in terms that are specific to each application. Yet our conditions and conclusions apply to all of them at a time. This is because we have arrived at the abstract formulation of our environments by scrutinizing what is common in the nature of these settings, and many others, for which results about ex post incentive compatibility and related concepts had been carefully explored.

A second reason is that, in each of the applications, we can provide blood and flesh to the general and rather abstract notion of a preference function, by exhibiting how it is defined to fit the particulars of the case at hand.

A third and very important reason to present the examples in pairs is because they allow us to show that the frontier between worlds where impossibility prevails, and others where ex post incentive compatibility is compatible with a high degree of efficiency can be surprisingly thin. For each one of our fields of application, we present examples that look rather similar and yet belong to one of these worlds or to the other, depending on whether the preference function associated with a set of types leads to a knit environment or does not. Since knit environments are also partially knit, our Theorem 2 applies also there, if only to the constant function. But our examples clarify that attractive mechanisms may exist on partially knit environments.

The paper proceeds as follows. In the next Section 2 we present the general framework and define the restrictions on environments that we propose, and the kind of mechanisms we shall concentrate on. Section 3 contains the general results and their proofs. Section 4 provides examples of applications and ties them in with our general framework. Appendix A and B contain proofs of results presented in Section 2 and 4, respectively.

[3]A pioneering paper by Shenker (1993) investigated the connections between individual and group strategy-proof non-bossy social choice rules in economic environments. For a recent reference on efficiency in general environments, see Copic (2017).

2 The model

Let $N = \{1, 2, ..., n\}$ be a finite set of *agents* with $n \geqslant 2$ and A be a set of *alternatives.*

Each agent $i \in N$ is endowed with a *type* θ_i belonging to a set Θ_i. Each θ_i includes all the information in the hands of i. We denote by $\Theta = \times_{i \in N} \Theta_i$ the set of type profiles. A *type profile* is an n-tuple $\theta = (\theta_1, ..., \theta_n) \in \Theta$ that we will write as $\theta = (\theta_C, \theta_{N \backslash C})$ when we want to stress the role of coalition C in N.

Let $\widetilde{\mathcal{R}}$ be the set of all complete, reflexive, and transitive binary relations on A and $\mathcal{R}_i \subseteq \widetilde{\mathcal{R}}$ be the set of those preferences that are allowed for individual i. While $R_i \in \widetilde{\mathcal{R}}$ denotes agent i's preferences, let P_i and I_i be the strict and the indifference part of R_i, respectively.

Once type profiles are fully determined, so are the agent's preference profiles. We formalize this dependence through the notion of a *preference function.*

Definition 1 *Let Θ be a set of types. A* ***preference function*** *R on Θ, $R : \Theta \to \times_{i \in N} \mathcal{R}_i$, assigns a preferences profile $R(\theta)$ to each type profile $\theta \in \Theta$.*

We call $R(\theta) = (R_1(\theta), ..., R_n(\theta))$ the preferences profile induced by the type profile θ while $R_i(\theta) \in \mathcal{R}_i$ stands for the induced preferences of agent i at θ. Notice that $\mathcal{R}_i$ may be different for each agent.[4] As usual $P_i(\theta)$ and $I_i(\theta)$ denote the strict and the indifference part of $R_i(\theta)$, respectively. Note that the domain of the preference function R is a Cartesian product including all possible type profiles, but its range may be a non-Cartesian strict subset of $\times_{i \in N} \mathcal{R}_i$.

An *environment* is a pair (Θ, R) formed by a set of types and a preference function. Following standard use, *private values environments* are those where each agent's component of the preference function only depends on her type. That is, $R_i(\theta) = R_i(\theta_i, \theta'_{N \backslash \{i\}})$ for each agent $i \in N$, $\theta \in \Theta$, and $\theta'_{N \backslash \{i\}} \in \times_{j \in N \backslash \{i\}} \Theta_j$. Otherwise, we are in *interdependent values environments.* In private values environments, abusing notation, we will write $R_i(\theta_i)$ instead of $R_i(\theta)$.

In some private values environments, individual types can be identified with their individual preferences. These are those where the preference function is biunivocal and establishes a one to one relationship between an agent's type and this agent's component of the preference function. Then, we can identify the environment with the set of preference profiles and properly refer to the constraints on environments as domain restrictions.

Elements in the range of a preference function may be restricted to satisfy further conditions. In particular, if an environment (Θ, R) is such that for all $\theta \in \Theta$ and agent $i \in N$, $R_i(\theta) \in \mathcal{R}_i$ is a strict preference, we will say that this environment is *strict.*

Our results refer to direct mechanisms. In fact, the properties we discuss are best analyzed with reference to the direct mechanism associated to any general one that might be described in terms of different message spaces and outcome functions.

A *direct mechanism on* Θ is a function $f : \Theta \to A$ such that $f(\theta) \in A$ for each $\theta \in \Theta$. From now on, we drop the term "direct" and refer to mechanisms, without danger of ambiguity.

[4] This is the case, for example, in economies with private goods when individuals are selfish.

Notice that, by letting Θ be the domain of f, we implicitly assume that all type profiles within this set are considered to be feasible by the designer.

We shall now identify two important conditions on environments (Definitions 5 and 6) that may or may not be satisfied. Both conditions start by considering sequences of type profiles that result from changing the type of individual agents, one at a time. These sequences are identified in detail in Definitions 5 and 6. Before that, we need some previous notation and definitions.

For any $x \in A$ and $R_i \in \widetilde{\mathcal{R}}$, $U(R_i, x) = \{y \in A : yR_ix\}$ is the *upper contour set of* R_i *at* x and $\overline{U}(R_i, x) = \{y \in A : yP_ix\}$ is the *strict upper contour set of* R_i *at* x. It will be useful to pay attention to the relationship between certain pairs of preferences.

Definition 2 *We say that* $R_i' \in \widetilde{\mathcal{R}}$ *is an* x-***monotonic transform of*** $R_i \in \widetilde{\mathcal{R}}$ *if* $U(R_i', x) \subseteq U(R_i, x)$ *and* $\overline{U}(R_i', x) \subseteq \overline{U}(R_i, x)$.

Equivalently, R_i' is an x-monotonic transform of R_i if there exists a subset of x's indifference class in R_i, containing x, such that the relative position of its elements has weakly improved when going from R_i to R_i'.[5]

A special class of monotonic transforms that are easy to identify are those where two preference relations have exactly the same weak and strict upper contour sets for a given alternative x. Then we say that they are *reshufflings* of each other, and each of the two preferences are, in particular, monotonic transforms of the other.

We are now able to define the sequences of types relevant in the definitions of knit and of partially knit environments.

Let $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$ be a sequence of individual types of length t_S, such that for each $h \in \{1, ..., t_S\}$, $\theta^S_{i(S,h)} \in \Theta_{i(S,h)}$. Agents may appear in that sequence several times or not at all. $I(S) = \{i(S, 1), ..., i(S, t_S)\}$ is the sequence of agents whose types appear in S and $i(S, h)$ is the agent in position h in S.

Given $\theta \in \Theta$ and $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$, we consider the sequence of type profiles $m^h(\theta, S)$ that results from changing one at a time the types of agents according to S, starting from θ. Formally, $m^h(\theta, S) \in \Theta$ is defined recursively so that $m^0(\theta, S) = \theta$ and for each $h \in \{1, ..., t_S\}$, $m^h(\theta, S) = \left(\left(m^{h-1}(\theta, S)\right)_{N \setminus i(S,h)}, \theta^S_{i(S,h)}\right)$.

Definition 3 *Let* $\theta \in \Theta$, *and* $S = \left\{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\right\}$. *We call the sequence of type profiles* $\left\{m^h(\theta, S)\right\}_{h=0}^{t_S}$ ***the passage from*** θ ***to*** θ' ***through*** S *if* $m^{t_S}(\theta, S) = \theta'$ *for* $\theta' \in \Theta$.

More informally, we say that θ leads to θ' through S.

Notice that a given passage from θ to θ' through S induces a corresponding sequence of preference profiles, $R^h(\theta, S) = (R^h_1(\theta, S), ..., R^h_n(\theta, S)) \in \times_{i \in N} \mathcal{R}_i$ for $h \in \{0, 1, ..., t_S\}$ where for each agent $i \in N$, we define $R^h_i(\theta, S) \equiv R_i\left(m^h(\theta, S)\right) \in \mathcal{R}_i$, that is, as the ith component of the preference function at the type profile $m^h(\theta, S)$.

[5] In our previous paper Barberà, Berga, and Moreno (2012), we present a similar condition but with additional requirements.

We can now establish a condition on the connection between sequences of changes in type profiles and the changes in preference profiles that they induce by means of the preference function.

Definition 4 *Let $x \in A$, $\theta, \theta' \in \Theta$. We will say that the passage from θ to θ' through S is x-**satisfactory** if for each $h \in \{1, ..., t_S\}$, $R^{h}_{i(S,h)}(\theta, S)$ is an x-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.*

Notice that in the case of private values the order of individuals in S could be changed and the new sequence would still serve the same purpose. This is because the changes in the type of each agent only induce changes in the preferences of this agent. By contrast, the precise order of agents $I(S)$ may be crucial in the case of interdependent values. We say that x is the reference alternative when going from θ to θ'.

We use Example 1 to illustrate the concept of satisfactory and non-satisfactory passages in an interdependent values environment.[6]

Example 1 *Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Each agent i has two possible types: $\Theta_i = \{\underline{\theta}_i, \overline{\theta}_i\}$. The preference function R is defined in Table 1. We write, in each cell, the preferences of both agents for a given type profile represented by an ordered list from better to worse, with parethesis in case of indifferences. Observe that agent 2's preferences over b and c depend on agent 1's type: $bP_2(\underline{\theta}_1, \underline{\theta}_2)c$ while $cP_2(\overline{\theta}_1, \underline{\theta}_2)b$, that is, we are in an interdependent values environment.*

R	$\underline{\theta}_2$		$\overline{\theta}_2$	
$\underline{\theta}_1$	$R_1(\underline{\theta}_1, \underline{\theta}_2)$ *acb*	$R_2(\underline{\theta}_1, \underline{\theta}_2)$ *b(ac)*	$R_1(\underline{\theta}_1, \overline{\theta}_2)$ *bca*	$R_2(\underline{\theta}_1, \overline{\theta}_2)$ *a(bc)*
$\overline{\theta}_1$	$R_1(\overline{\theta}_1, \underline{\theta}_2)$ *c(ab)*	$R_2(\overline{\theta}_1, \underline{\theta}_2)$ *c(ab)*	$R_1(\overline{\theta}_1, \overline{\theta}_2)$ *c(ab)*	$R_2(\overline{\theta}_1, \overline{\theta}_2)$ *c(ab)*

Table 1. Preference function for Example 1.

Notice that the range of R is not a Cartesian product, since $\mathcal{R}_1 = \{acb, bca, c(ab)\}$ and $\mathcal{R}_2 = \{b(ac)), a(bc), c(ab)\}$ but the preferences profile $(acb, a(bc))$ are not in the range of the preference function R.

Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \underline{\theta}_2)$, and $S = \left\{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\right\}$ a sequence of individual types. Note that, $I(S) = \{2, 1, 2\}$ and $t_S = 3$. We claim that the passage from θ to θ' through S is a-satisfactory. To show it, we have to check that for each $h \in \{1, 2, t_S = 3\}$, $R^{h}_{i(S,h)}(\theta, S)$ is an a-monotonic transform of $R^{h-1}_{i(S,h)}(\theta, S)$.

For that, observe first that $R^{0}_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \underline{\theta}_2)$, $R^{1}_{i(S,1)}(\theta, S) = R_2(\underline{\theta}_1, \overline{\theta}_2)$, $R^{1}_{i(S,2)}(\theta, S) = R_1(\underline{\theta}_1, \overline{\theta}_2)$, $R^{2}_{i(S,2)}(\theta, S) = R_1(\overline{\theta}_1, \overline{\theta}_2)$, $R^{2}_{i(S,2)}(\theta, S) = R_2(\overline{\theta}_1, \overline{\theta}_2)$, and $R^{3}_{i(S,2)}(\theta, S) = R_2(\overline{\theta}_1, \underline{\theta}_2)$. Then, using the table in Example 1, note that the following three facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2) =$

[6] This example adapts, in ordinal terms, the one proposed by Bergemann and Morris (2005) as their Example 1.

$a(bc)$ *is an* a*-monotonic transform of* $R_2(\underline{\theta}_1, \underline{\theta}_2) = b(ac)$ *since* $U(R_2(\underline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\} \subseteq U(R_2(\underline{\theta}_1, \underline{\theta}_2), a) = \{a, b, c\}$ and $\overline{U}(R_2(\underline{\theta}_1, \overline{\theta}_2), a) = \varnothing \subseteq \overline{U}(R_2(\underline{\theta}_1, \underline{\theta}_2), a) = \{b\}$. *Moreover,* $R_1(\overline{\theta}_1, \overline{\theta}_2) = c(ab)$ *is an* a*-monotonic transform of* $R_1(\underline{\theta}_1, \overline{\theta}_2) = bca$ since *since* $U(R_1(\overline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\} \subseteq U(R_1(\underline{\theta}_1, \overline{\theta}_2), a) = \{a, b, c\}$ and $\overline{U}(R_1(\overline{\theta}_1, \overline{\theta}_2), a) = \{c\} \subseteq \overline{U}(R_1(\underline{\theta}_1, \overline{\theta}_2), a) = \{b, c\}$. *Finally,* $R_2(\overline{\theta}_1, \underline{\theta}_2) = c(ab)$ *is an* a*-reshuffling of* $R_2(\overline{\theta}_1, \overline{\theta}_2) = c(ab)$ since both preferences coincide.
Let $x = a$, $\theta = (\underline{\theta}_1, \underline{\theta}_2)$, $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, *and* $S = \left\{\overline{\theta}_1, \overline{\theta}_2\right\}$ *a sequence of individual types. Note that,* $I(S) = \{1, 2\}$ *and* $t_S = 2$. *We claim that the passage from* θ *to* θ' *through* S *is not* a*-satisfactory. To show it, observe that for* $h = 1$, $R^h_{i(S,h)}(\theta, S)$ *is not an* a*-monotonic transform of* $R^{h-1}_{i(S,h)}(\theta, S)$. *By definition,* $R^0_{i(S,1)}(\theta, S) = R_1(\theta)$ and $R^1_{i(S,1)}(\theta, S) = R_1(\overline{\theta}_1, \underline{\theta}_2)$. *Moreover,* $R_1(\overline{\theta}_1, \underline{\theta}_2) = c(ab)$ *is not an* a*-monotonic transform of* $R_1(\theta) = acb$ *since* $\overline{U}(R_1(\overline{\theta}_1, \underline{\theta}_2), a) = \{a, b, c\} \nsubseteq U(R_1(\theta), a) = \{a\}$ (in fact, $\overline{U}(R_1(\overline{\theta}_1, \underline{\theta}_2), a) = \{c\} \nsubseteq \overline{U}(R_1(\theta), a) = \varnothing$).

Armed with our previous definitions we now identify our first restriction on environments.

Definition 5 *We say that an environment* (Θ, R) *is* ***knit*** *if for any two pairs formed by an alternative and a type profile each,* (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$, $\theta \neq \widetilde{\theta}$, $x \neq z$, *there exist* $\theta' \in \Theta$ *and sequences of types* S *and* $\widetilde{S}$, *such that the passage from* θ *to* θ' *through* S *is* x*-satisfactory and the passage from* $\widetilde{\theta}$ *to* θ' *through* $\widetilde{S}$ *is* z*-satisfactory.*

Two important remarks are in order. First, whether or not an environment is knit will depend on the way how the preference function determines what sequences are considered to be satisfactory. Moreover, when going through proofs of knitness (see Remark 1, for example) the reader can observe that for some pairs formed by an alternative and a type profile each, there exist several type profiles and passages that work. Knitness requires only the existence of one such way. The following remark provides an example of how to check whether an environment is knit.

Remark 1 *The environment in Example 1 is knit.*

To check that the environment (Θ, R) where $\Theta = \left\{(\underline{\theta}_1, \underline{\theta}_2), (\underline{\theta}_1, \overline{\theta}_2), (\overline{\theta}_1, \underline{\theta}_2), (\overline{\theta}_1, \overline{\theta}_2)\right\}$ is knit, we must prove that all pairs of alternatives and types can be connected through satisfactory sequences. To do that, we will show how to choose the appropriate ones for two specific cases, and then argue that all others can be reduced essentially to one of the patterns we shall follow.
Case 1. $(x, \theta) = (a, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (b, (\overline{\theta}_1, \underline{\theta}_2))$.
Define $\theta' = \widetilde{\theta} = (\overline{\theta}_1, \underline{\theta}_2)$, $S = \left\{\overline{\theta}_2, \overline{\theta}_1, \underline{\theta}_2\right\}$ (thus, $I(S) = \{2, 1, 2\}$ and $t_S = 3$), $\widetilde{S} = \varnothing$ (thus, $I(\widetilde{S}) = \varnothing$ and $t_{\widetilde{S}} = 0$). Note that since $\theta' = \widetilde{\theta}$, then $\widetilde{\theta}$ trivially leads to θ' through $\widetilde{S}$ and this passage from $\widetilde{\theta}$ to θ' is b-satisfactory. We need to show that θ leads to θ' through S and the passage is a-satisfactory. For that we need to observe using Table 1 that the three (t_S) following facts hold: $R_2(\underline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_2(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$. Finally, $R_2(\overline{\theta}_1, \underline{\theta}_2)$ is an a-reshuffling of $R_2(\overline{\theta}_1, \overline{\theta}_2)$.

Case 2. $(x, \theta) = (c, (\underline{\theta}_1, \underline{\theta}_2))$ and $(z, \widetilde{\theta}) = (a, (\underline{\theta}_1, \overline{\theta}_2))$.
Define $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), $\widetilde{S} = \{\overline{\theta}_1\}$ (thus, $I(\widetilde{S}) = \{1\}$ and $t_{\widetilde{S}} = 1$). As above, first we need to show that θ leads to θ' through S and the passage is a-satisfactory. For that we need to observe using Table 1 that the two (t_S) following facts hold: $R_1(\overline{\theta}_1, \underline{\theta}_2)$ is a c-monotonic transform of $R_1(\underline{\theta}_1, \underline{\theta}_2)$. Moreover, $R_2(\overline{\theta}_1, \overline{\theta}_2)$ is a c-reshuffling of $R_2(\overline{\theta}_1, \underline{\theta}_2)$.
Second, we need to show that $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$ and the passage is a-satisfactory. For that we need to observe using the table that $R_1(\overline{\theta}_1, \overline{\theta}_2)$ is an a-monotonic transform of $R_1(\underline{\theta}_1, \overline{\theta}_2)$.
To finish the proof of knitness we should consider all remaining combinations of (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. Observe that each one of those cases can be embedded in either Case G1 or Case G2 below, which generalize Cases 1 and 2, respectively.
Case G1. (x, θ) and $(z, \widetilde{\theta})$ such that $x \in \{a, b\}$.
Case G2. (x, θ) and $(z, \widetilde{\theta})$ such that $x = c$.
To prove knitness for Case G1, consider $\theta' = \widetilde{\theta}$, $\widetilde{S} = \varnothing$, and S will depend on θ and $\widetilde{\theta}$. Similarly, to prove knitness for Case G2, consider $\theta' = (\overline{\theta}_1, \overline{\theta}_2)$, $S = \{\overline{\theta}_1, \overline{\theta}_2\}$ (thus, $I(S) = \{1, 2\}$ and $t_S = 2$), and $\widetilde{S}$ will depend on θ and $\widetilde{\theta}$.

We shall now define partially knit environments. This condition is less demanding that knitness because it only requires to connect some pairs of type profiles, and only for some pairs of reference alternatives. Whether or not an environment is partially knit will again depend on how the preference function determines what sequences are satisfactory, but now the pairs of type profiles and alternatives involved will be more limited.

For any $\theta \in \Theta$ and $x, z \in A$, let $C(\theta, z, x) = \{i \in N : zR_i(\theta)x\}$ and $\overline{C}(\theta, z, x) = \{j \in N : zP_j(\theta)x\}$.

Definition 6 *We say that an environment (Θ, R) is* ***partially knit*** *if for any two pairs formed by an alternative and a type profile each, (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$, $\theta \neq \widetilde{\theta}$, such that $\overline{C}(\theta, z, x) \neq \emptyset$, $\#C(\theta, z, x) \geq 2$, and $\widetilde{\theta}_j = \theta_j$ for any $j \in N \backslash C(\theta, z, x)$, then there exist $\theta' \in \Theta$ and sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.*

Clearly, if an environment is knit it is also partially knit. Observe that since $\overline{C}(\theta, z, x) \neq \varnothing$, then $z \neq x$.

Notice that, here again, partial knitness is satisfied as long as there is one satisfactory passage for each relevant pair of alternatives and profiles.

A number of consequences of our definitions for private values environments follow. We start by the remark that essentially no such environment will be knit. The proof of Proposition 1 is found in Appendix A.

Proposition 1 *No private values environment (Θ, R) for which there exist $\theta_i, \widetilde{\theta}_i \in \Theta_i$ such that $R_i(\theta_i) \neq R_i(\widetilde{\theta}_i)$ for some $i \in N$ can be knit.*

Propositions 2, 3, and 4 discuss the pertinence of our new properties for several important private values environments where the preference function is biunivocal.

We begin by the universal domain of strict preferences. The proof of Proposition 2 is found in Appendix A.

Proposition 2 *The set of all strict preferences in the classical social choice problem is partially knit.*

Another interesting case is provided by the set of strict single-peaked preferences on a finite set of alternatives. We know that it is not knit by Proposition 1, but as stated in Proposition 3 and proven in Appendix A, it is partially knit.

Proposition 3 *The set of all strict single-peaked preferences on a finite set of alternatives in the classical social choice problem is partially knit.*

In the housing problem, agents' admissible preferences over their individual assignment are strict. And, again, they define a partially knit environment, as stated in Proposition 4 and proven in Appendix A.

Proposition 4 *The set of preferences in the housing problem is partially knit.*[7]

Until now, we have concentrated on the properties of potential environments. We now turn attention to some properties of the mechanisms themselves.

We first look at incentives. *Ex post incentive compatibility* requires, for all agents to prefer truthtelling at a given type profile θ, if all the other agents also report truthfully.[8] Therefore, truthful revelation is required to be a Nash equilibrium.

Definition 7 *Let* (Θ, R) *be an environment. A mechanism* f *is* ***ex post incentive compatible*** *in* (Θ, R) *if, for all agent* $i \in N$, $\theta \in \Theta$, *and* $\theta_i' \in \Theta_i$, $f(\theta) R_i(\theta) f(\theta_i', \theta_{N\setminus\{i\}})$.[9]

We say that an agent $i \in N$ *can ex post profitably deviate under mechanism* f *at* $\theta \in \Theta$ if there exists $\theta_i' \in \Theta_i$ such that $f(\theta_i', \theta_{N\setminus\{i\}}) P_i(\theta) f(\theta)$. Note that ex post incentive compatibility requires that no agent can profitably deviate at any type profile.

In addition to individuals, coalitions of agents may also jointly deviate if they find it profitable. This leads us to propose the following definition.

Definition 8 *Let* (Θ, R) *be an environment. We say that* ***a coalition*** $C \subseteq N$ ***can ex post profitably deviate under mechanism*** f ***at*** $\theta \in \Theta$ *if there exists* $\theta_C' \in \times_{i \in C} \Theta_i$ *such that for all agent* $i \in C$, $f(\theta_C', \theta_{N\setminus C}) R_i(\theta) f(\theta)$ *and for some* $j \in C$, $f(\theta_C', \theta_{N\setminus C}) P_j(\theta) f(\theta)$. *A mechanism* f *is* ***ex post group incentive compatible*** *in* (Θ, R) *if no coalition of agents can ex post profitably deviate at any type profile.*[10]

[7] The same result would hold in the one-to-one matching problem where admissible preferences over individual assignments are strict and different for each agent: those of each woman are defined on all men and on herself, while those of each man are defined on all women and himself.

[8] This property is called uniform incentive compatibility by Holmstrom and Myerson (1983). See also Bergemann and Morris (2005).

[9] From now on we omit reference to the environments on properties of f when no confusion arises.

[10] Notice that we allow for some agents to participate in the profitable deviation without strictly gaining from it. Moreover, we also allow for some agents not to change their types. That facilitates the deviation by groups. Therefore, our requirement of ex post group incentive compatibility is strong.

Finally, we may require our mechanisms to satisfy a condition that we call respectfulness. This is a condition similar to those imposed in the literature when dealing with environments where agents' preferences allow for non-degenerate indifference classes (See Thomson, 2016). Relative to other technical conditions of the same sort, ours is among the weakest, because it only applies to some limited changes in type profiles, and has no bite in some important cases (for example, in public good economies where agents' preferences are strict). The condition essentially demands that for those specific changes in type profiles, no agent should affect the outcome (for her and for others) unless she changes her level of satisfaction.

Definition 9 *Let* (Θ, R) *be an environment. A mechanism* f *is* ***(outcome) respectful*** *in* (Θ, R) *if*

$$f(\theta) I_i(\theta) f(\theta'_i, \theta_{N\setminus\{i\}}) \text{ implies } f(\theta) = f(\theta'_i, \theta_{N\setminus\{i\}}),$$

for each $i \in N$, $\theta \in \Theta$, *and* $\theta'_i \in \Theta_i$ *such that* $R_i(\theta'_i, \theta_{N\setminus\{i\}})$ *is a* $f(\theta)$*-monotonic transform of* $R_i(\theta)$.[11]

For short, we call this condition respectfulness.

The following two Paretian notions of efficiency will be used in our discussion of results.

Definition 10 *Let* (Θ, R) *be an environment. A mechanism* f *is* ***Pareto efficient on the range*** *in* (Θ, R) *if for all* $\theta \in \Theta$, *there is no alternative* x *in the range of* f *such that* $xR_i(\theta)f(\theta)$ *for all* $i \in N$ *and* $xP_j(\theta)f(\theta)$ *for some* $j \in N$. *If, in addition, the mechanism is onto* A *we say that it is* ***fully efficient*** *in* (Θ, R).

Notice that ex post group incentive compatibility implies Pareto efficiency on the range, since otherwise the grand coalition could profitably deviate.

3 The results

Our first result shows that only constant mechanisms can be ex post incentive compatible and respectful in knit environments. Before we prove the theorem, let's comment on its importance and implications. The conclusion of Theorem 1 is very strong, and it is in the same vein than the one in Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006) obtain under completely different premises. The theorem also restricts attention to mechanisms that are respectful, but note that the latter requirement does not always have bite: It is irrelevant when the environment is strict, that is, when the preferences of all agents under all type profiles are strict (see Corollary 1 below). Also observe that since we work with functions, our environments are separable, in the sense of Bergemann and Morris (2005) who also show (see their Proposition 2) that in this case only rules that are ex post incentive compatible could be interim incentive compatible. Therefore, our theorem also applies for the latter weaker requirement, whatever the priors of agents might be, and with no need to be specific about them.

[11] Respectfulness is an analogue condition to the one we use in Barberà, Berga, and Moreno (2016) but requiring here invariance in outcomes instead of indifferences in outcomes. Examples of mechanisms satisfying respectfulness are provided in Section 4. An example of a mechanism violating it is the Gale-Shapley mechanism (see Barberà, Berga, and Moreno, 2016).

Theorem 1 *Let (Θ, R) be a knit environment and $f : \Theta \to A$ be a mechanism. If f is ex post incentive compatible and respectful, then f is constant.*[12]

Corollary 1 *Let (Θ, R) be a strict knit environment and $f : \Theta \to A$ be a mechanism. If f is ex post incentive compatible, then f is constant.*

The proof of Corollary 1 is obtained using the first part in each step of the proof of Theorem 1 where respectfulness is not used. Some standard notation is required for the proof: For any $x \in A$ and $R_i \in \widetilde{\mathcal{R}}$, $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$ is the *strict lower contour set of R_i at x* and $E(R_i, x) = \{y \in A : yI_ix\}$ is the *indifference class of R_i at x*.

Proof of Theorem 1. Let (Θ, R) be a knit environment and let f be an ex post incentive compatible and respectful mechanism. Assume, by contradiction, that f was not constant. Then, there will be $x, z \in A$, $x \neq z$ such that $x = f(\theta)$ and $z = f(\widetilde{\theta})$ for some θ and $\widetilde{\theta}$ in Θ. Since (Θ, R) is knit, for the two pairs formed by an alternative and a type profile, (x, θ) and $(z, \widetilde{\theta}) \in A \times \Theta$, there exist $\theta' \in \Theta$ and two sequences $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Now, we will show the following:
(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and
(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$.
Statements in (a) and (b) yield to a contradiction. By definition of the sequences S and $\widetilde{S}$, we know that $m^{t_S}(\theta, S) = m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S}) = \theta'$. However, $f(\theta') = f(m^{t_S}(\theta, S)) = x$ by (a) while $f(\theta') = f(m^{t_{\widetilde{S}}}(\widetilde{\theta}, \widetilde{S})) = z$ by (b).
We prove (a) in steps, from $h = 1$ to $h = t_S$. The proof of (b) is identical and omitted.
Step 1. Let $h = 1$. By Definition 4, $R^1_{i(S,1)}(\theta, S)$ is an x-monotonic transform of $R^0_{i(S,1)}(\theta, S) = R_{i(S,1)}(\theta)$. (1)
Observe that $f(m^1(\theta, S)) \notin \overline{L}\left(R^1_{i(S,1)}(\theta, S), x\right)$. (2)
(otherwise, if $f(m^1(\theta, S)) \in \overline{L}\left(R^1_{i(S,1)}(\theta, S), x\right)$, we would get a contradiction to ex post incentive compatibility since $i(S, 1)$ would ex post profitably deviate under f at $(\theta^S_{i(S,1)}, (m^0(\theta, S))_{N\setminus\{i(S,1)\}})$ via $\theta_{i(S,1)}$).
By (1) and (2) we have that $f(m^1(\theta, S)) \notin \overline{L}\left(R^0_{i(S,1)}(\theta, S), x\right)$. (3)
By ex post incentive compatibility of f, $f(m^1(\theta, S)) \notin \overline{U}\left(R^0_{i(S,1)}(\theta, S), x\right)$. (4)
(otherwise, if $f(m^1(\theta, S)) \in \overline{U}\left(R^0_{i(S,1)}(\theta, S), x\right)$, $f(m^1(\theta, S))P^0_{i(S,1)}(\theta)x$ contradicting ex post incentive compatibility since $i(S, 1)$ would ex post profitably deviate under f at θ via $\theta^S_{i(S,1)}$).
Thus, by (3) and (4) we have that $f(m^1(\theta, S)) \in E\left(R^0_{i(S,1)}(\theta, S), x\right)$. (5)
Then, by respectfulness, we get that $f(m^1(\theta, S)) = f(m^0(\theta, S)) = f(\theta) = x$ which ends the

[12] In a companion paper Barberà, Berga, and Moreno (2018), we show that for the case of two alternatives at stake, knitness is not only sufficient for obtaining that ex post incentive compatible and respectful mechanisms be constant, but it is also necessary.

proof of (a) for $h = 1$.
Step $h \in \{2, ..., t_S\}$. By repeating the same argument than in Step 1 on the recursive fact that $f(m^{h-1}(\theta, S)) = x$, we obtain that $f(m^h(\theta, S)) = f(m^{h-1}(\theta, S)) = x$. ■

We now prove our second result, showing the equivalence between ex post individual and group incentive compatibility in partially knit environments. This result has bite for both private and interdependent values environments.

Theorem 2 *Let (Θ, R) be a partially knit environment and f be a respectful mechanism. Then, f is ex post incentive compatible if and only if f is ex post group incentive compatible.*

Corollary 2 *Let (Θ, R) be a strict partially knit environment and f be a mechanism. Then, f is ex post incentive compatible if and only if f is ex post group incentive compatible.*

Before we prove the theorem, let us discuss its content and implications.

A first consequence of ex post group incentive compatibility is Pareto efficiency on the mechanism's range. Hence, the implications that having a good performance regarding incentives may be compatible with efficiency is an invitation to investigate those cases where this may be a promising possibility.

Also observe that in private values cases where environments are partially knit (see Propositions 2, 3, and 4, for example), the result in Theorem 2 admits a second reading. This is because ex post incentive compatibility then becomes equivalent to strategy-proofness,[13] since each agent i's preferences depend on θ only through θ_i. For the same reason, ex post group incentive compatibility becomes equivalent to strong group strategy-proofness. These remarks lead us to the following corollary.

Corollary 3 *Let (Θ, R) be a partially knit environment in private values and let f be a respectful mechanism. Then, f is strategy proof if and only if f is strongly group strategy-proof.*

The equivalence between individual and group ex post incentive compatibility may hold in rather vacuous ways, because there are cases where the only ex post incentive compatible rules lack any interest. But there are other cases where there is a real possibility of making these desiderata compatible in non-trivial ways.

Here are three relevant examples of mechanisms for which the result holds non-trivially in private values environments. One of them is the family of social choice functions defined on the set of all strict single-peaked preferences (see Moulin, 1980 and our Proposition 3). The other case is provided by the top trading cycle mechanism for house allocation (see Shapley and Scarf, 1974 and our Proposition 4). Finally, consider the large class of non-trivial strategy-proof rules on the set of all strict preferences that one can define when only

[13] We say that a mechanism f is weakly group manipulable at $\theta \in \Theta$ if there exist a coalition $C \subseteq N$ and $\theta'_C \in \times_{i \in C} \Theta_i$ ($\theta'_i \neq \theta_i$ for any $i \in C$) such that $f(\theta'_C, \theta_{-C}) R_i(\theta_i) f(\theta)$ for all $i \in C$ and $f(\theta'_C, \theta_{-C}) P_j(\theta_j) f(\theta)$ for some $j \in C$. A mechanism f is strongly group strategy-proof in an environment (Θ, R) if f is not weakly group manipulable at any $\theta \in \Theta$. When the condition is imposed only on singleton coalitions $C = \{i\}$, we say that f is strategy-proof (also called dominant strategy incentive compatible). In words, strategy-proofnes requires that all agents prefer truthtelling at a given type profile θ, whatever all the other agents report.

two alternatives are at stake (see Barberà, Berga, and Moreno, 2012 and Manjunath 2012). In all three cases we are dealing with partially knit private values environments where a type for an agent can be identified with her preference relation, the mechanisms are individual and strongly group strategy-proof, and by no means trivial. Also remark that for the case where the mechanism has more than two alternatives on the range, only dictatorship is strategy-proof on the universal set of preferences, by the Gibbard-Satterthwaite theorem (see Gibbard, 1973 and Satterthwaite, 1975). This is an example in which our Theorem 2 also applies, since the universal set of preferences is partially knit and dictatorships are strongly group strategy-proof, but we use it here as a warning sign that the implications of Theorem 2, as already explained may or may not be of interest depending on the environments.[14]

Proof of Theorem 2. Let (Θ, R) be a partially knit environment and let f be a respectful mechanism. By definition, ex post group incentive compatibility implies ex post incentive compatibility. To prove the converse, suppose, by contradiction, that there exist $\theta \in \Theta$, $C \subseteq N$, $\#C \geq 2$, $\widetilde{\theta}_C \in \times_{i \in C} \Theta_i$ such that for any agent $i \in C$, $f(\widetilde{\theta}_C, \theta_{N \setminus C}) R_i(\theta) f(\theta)$ and $f(\widetilde{\theta}_C, \theta_{N \setminus C}) P_j(\theta) f(\theta)$ for some agent $j \in C$. Let $z = f(\widetilde{\theta}_C, \theta_{N \setminus C})$ and $x = f(\theta)$. Note that (i) $z \neq x$, (ii) $\overline{C}(\theta, z, x) \neq \varnothing$, $\#C(\theta, z, x) \geq 2$ since $C \subseteq C(\theta, z, x)$, and (iii) $\widetilde{\theta}_j = \theta_j$ for any $j \in N \setminus C(\theta, z, x)$ again since $C \subseteq C(\theta, z, x)$.
Since (Θ, R) is partially knit and conditions in Definition 6 are satisfied, there exist $\theta' \in \Theta$ and two sequences of types $S = \{\theta^S_{i(S,1)}, ..., \theta^S_{i(S,t_S)}\}$, $\widetilde{S} = \{\widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},1)}, ..., \widetilde{\theta}^{\widetilde{S}}_{i(\widetilde{S},t_{\widetilde{S}})}\}$ such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Although these sequences are not necessarily the same than the ones we used in the proof of Theorem 1, from this point on, we can use the same reasoning as there, and show that
(a) for each $h \in \{1, ..., t_S\}$, $f(m^h(\theta, S)) = x$, and
(b) for each $h \in \{1, ..., t_{\widetilde{S}}\}$, $f(m^h(\widetilde{\theta}, \widetilde{S})) = z$,
again leading to a contradiction. Adding the arguments we have already used in the proof of Theorem 1 we would complete the one for the present theorem. ■

4 Applications

In this section we present examples of simple environments where our theorems apply.

These examples are inspired in our reading of several relevant papers in the literature. They are framed in the language we have developed in our paper, and they allow us to clarify several of the points we try to make all along.

Examples 2 and 3 refer to deliberative committees and are inspired by our reading of Austen-Smith and Feddersen (2006), who build on the classical Condorcet jury problem and add the possibility that agents share (true or false) information.

[14]Let us comment on the connection between our results and the Gibbard-Satterthwaite theorem. There is no contradiction between our result in Theorem 1 that only constant mechanisms are strategy-proof and that of the Gibbard-Satterthwaite theorem, since the universal set of preferences where the latter applies is not knit, as shown in Proposition 1, and thus Theorem 1 does not apply.

Examples 4 and 5 refer to house allocation problems and are this time inspired by the analysis of Che, Kim, and Kojima (2015), regarding the existence of Pareto efficient and ex-post incentive compatible mechanisms in that context.

Examples 6 and 7 refer to auctions and are inspired by some of the models in Dasgupta and Maskin (2000) and Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006).

Until now we have analyzed our environments in abstract terms, and have not discussed the origin of the preference function, which indicates what is the relevant preferences profile associated to each profile of types. In Example 1 adapted from Example 1 in Bergemann and Morris (2005), the preference function is proposed without any specific explanation regarding where this function comes from. Indeed, this is common to different general discussions of the issues we address here, and digging on the underlying reasons to predicate a given preference function is immaterial for the validity of our theorems. Notice, however, that whether or not an environment is knit, or partially knit, will depend on the preference function that applies in each case, for environments that are otherwise identical. Hence, it is interesting to know, for each application, whether or not the underlying phenomenon we want to model is adequately represented by a specific preference function.

In most applications, authors endow agents with a general utility function[15] that may depend on variables that reflect the agent's type and, in the interdependent values case, on other variables that correspond to the types of the rest of agents. Our general framework has departed from this formulation, since we stick to a purely ordinal framework and avoid the use of utility functions. This has allowed us to define restrictions on environment that transcend the details of any particular functional form and avoid questions of representability. Since in this section we want to get closer to well studied issues, we also become precise about the form of preference functions, based on the interpretation of each model. That will allow us to show that the choice of preference functions crucially determines whether an environment of application is knit, or partially knit, and has implications on the possibilities of design.

4.1 Deliberative Juries

Example 2. A three-person jury $N = \{1, 2, 3\}$ must decide over two alternatives: whether to acquit (A) or to convict (C) a defendant under a given mechanism. The defendant is either guilty (g) or innocent (i). Each juror j gets a signal $s_j = g$ or $s_j = i$.

Jurors's preferences arise from combining the different signals they obtain from the deliberation, according to their bias in favor of acquittal in view of their observed signals and of those declared by others. In this example, jurors are either high-biased (h) or low-biased (l). High-biased jurors (h) prefer to convict if and only if all other jurors declare the guilty signal and they have also observed it $(s = (g, g, g))$, whereas low-biased ones (l) prefer to convict if and only if they have observed the guilty signal or at least one other committee member has declared it $(s \neq (i, i, i))$.

Each juror j's type is $\theta_j = (b_j, s_j) \in \Theta_j = B \times S$ where $B = \{h, l\}$ and $S = \{g, i\}$. A type profile $\theta \in \Theta = (B \times S)^n$. Let CA denote the preference to convict rather than to acquit and AC be the converse order. The preference function is defined such that for each type

[15] The use of utility functions that represent the preferences of expected utility maximizers is especially useful to analyze incentive compatibility notions that involve uncertainty regarding the types.

profile $\theta \in \Theta$ and for each juror $j \in N$, $R_j(\theta)$ is as follows:

$$R_j\left((b_j, s_j), \theta_{N\setminus\{j\}}\right) = \left\{ \begin{array}{l} CA \text{ if either } b_j = h \text{ and } s = (g,g,g) \text{ or } b_j = l \text{ and } s \neq (i,i,i), \\ AC, \text{ otherwise.} \end{array} \right\}$$

The environment (Θ, R) in this example is knit. Hence we know by Theorem 1 that it will be impossible to design non-constant, ex post incentive compatible, and respectful mechanisms in such framework.

The proof that the environment is knit is in Proposition 5 in Appendix B. Here we simply provide the reader with some hints on the techniques that we use to check for our restrictions in this example and subsequent ones.[16]

To check knitness for a particular pair of types and alternatives, (A, θ) and $(C, \widetilde{\theta})$, we must show that there are passages to a third type profile θ' which are A-satisfactory from θ and C-satisfactory from $\widetilde{\theta}$, respectively.

Consider the following three type profiles, $\theta = (\theta_1, \theta_2, \theta_3) = ((l,g),(h,g),(l,i))$, $\widetilde{\theta} = (\widetilde{\theta}_1, \widetilde{\theta}_2, \widetilde{\theta}_3) = ((l,g),(h,g),(l,g))$ and $\theta' = (\theta'_1, \theta'_2, \theta'_3) = ((l,i),(h,i),(l,i))$. The profiles of preferences they induce are shown in Table 2.

$R(\theta) = R((l,g),(h,g),(l,i))$	$R(\widetilde{\theta}) = R((l,g),(h,g),(l,g))$	$R(\theta') = R((l,i),(h,i),(l,i))$
$C \quad A \quad C$	$C \quad C \quad C$	$A \quad A \quad A$
$A \quad C \quad A$	$A \quad A \quad A$	$C \quad C \quad C$

Table 2: Agents' preferences induced by θ, $\widetilde{\theta}$, and θ', respectively.

As shown in Table 3, it is possible to sequentially move from θ to θ' by successively changing, one by one, the type of the agents as follows. First, agent 1 from (l,g) to (h,i), then agent 2 from (h,g) to (h,i) and finally agent 1 from (h,i) to (l,i). According to our notation, $I(S) = \{1,2,1\}$. Likewise, as shown in Table 4, we can move from $\widetilde{\theta}$ to θ' by successively changing, one by one, the type of some agents. First, agent 1, then agent 3 and finally agent 2, all from signal g to i, while their b's remain fixed. That is, $I(\widetilde{S}) = \{1,3,2\}$. In Table 3, alternative A either does not change its relative position (an A-reshuffling), or improves it (an A-monotonic transform). Similarly, in Table 4, the same requirements are satisfied but this time for alternative C.

$R(\theta) = R((l,g),(h,g),(l,i))$	$R((\mathbf{h},\mathbf{i}),(h,g),(l,i))$	$R((h,i),(h,\mathbf{i}),(l,i))$	$R(\theta') = R((\mathbf{l},i),(h,i),(l,i)$
$C \quad A \quad C$	$A \quad A \quad C$	$A \quad A \quad A$	$A \quad A \quad A$
$A \quad C \quad A$	$C \quad C \quad A$	$C \quad C \quad C$	$C \quad C \quad C$

Table 3: Induced agents' preferences given the specified type changes from θ to θ'.

[16] The reader that finds the following argument useful to better understand our condition may also find a similar one for partially knit in the text preceding the proof of Proposition 6 in Appendix B.

$R(\widetilde{\theta}) = R((l,g),(h,g),(l,g))$	$R((l,\mathbf{i}),(h,g),(l,g))$	$R((l,i),(h,g),(l,\mathbf{i}))$	$R(\theta') = R((l,i),(h,\mathbf{i}),(l,i)$
$C \quad C \quad C$	$C \quad A \quad C$	$C \quad A \quad C$	$A \quad A \quad A$
$A \quad A \quad A$	$A \quad C \quad A$	$A \quad C \quad A$	$C \quad C \quad C$

Table 4: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to θ'.

Example 3. Consider the framework of Example 2 and change the jurors' attitude to convict versus acquit as follows. Each juror may now be either unswerving or median. Unswerving jurors (u) prefer to convict if and only if they have observed the guilty sign and have also received such a sign from at least another juror. Median jurors (m) again prefer to convict under the same circumstances but also if they receive two guilty signals from other jurors.

For instance, if juror 1 is unswerving she will prefer to convict if either (g,g,g), (g,g,i), or (g,i,g) but if juror 2 is unswerving she will convict if either (g,g,g), (g,g,i), or (i,g,g)). Yet being median is the same for both agents, they will prefer to convict if either (g,g,g), (g,g,i), (g,i,g), or (i,g,g).

Each juror j's type is $\theta_j = (b_j, s_j) \in \Theta_j = B \times S$ where $B = \{u, m\}$ and $S = \{g, i\}$. A type profile $\theta \in \Theta = (B \times S)^n$. The preference function is defined such that for each type profile θ and for each juror $j \in N$, $R_j(\theta)$ is as follows:

$$R_j\left((b_j, s_j), \theta_{N\setminus\{j\}}\right) = \left\{ \begin{array}{ll} CA & \text{if either } b_j = u,\ s_j = g \text{ and } s_l = g \text{ for some } l \neq j, \\ & \text{or } b_j = m \text{ and } \#\{l \in N : s_l = g\} \geq 2, \text{ and} \\ AC & \text{otherwise.} \end{array} \right\}$$

This environment (Θ, R) is partially knit (see Proposition 6 in Appendix B) but not knit. To show that it is not knit, we present a family of mechanisms, the quota rules, that are non-constant, respectful, and ex post incentive compatible in (Θ, R).[17]

Let $q \in \{1, 2, 3\}$. A *voting by quota q* mechanism, f, chooses C for a type profile θ if and only if at least q agents have induced preferences from θ such that C is preferred to A.[18] Formally, for each type profile $\theta = (b, s) \in \Theta$,

$$f(\theta) = C \text{ if and only if } \#\{i \in N : R_i(\theta) = CA\} \geq q.$$

In Table 5 below we describe all possible results of voting by quota for different values of q in Example 3. We have four matrices, one for each type of agent 3. In the rows of each matrix we write the four types of agent 1 and in the columns the four types of agent 2. In each cell, we write each agent's best alternative according to their preferences at a given type profile, followed by the outcome of a quota mechanism. When two outcomes appear in a cell, the one in the left stands for the outcome of voting by quota 3 and the right one is the outcome for both quota 1 and 2, which in this example are always the same.

[17] Note that respectfulness is trivially satisfied in these environments where preferences are strict and alternatives have no private component.

[18] See Austen-Smith and Feddersen (2006) and Barberà and Jackson (2004) for papers where these rules are analized.

Given Table 5, it is easy to check that these rules are ex post incentive compatible. In addition, they also satisfy anonymity.

Now, Theorem 2 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible (therefore, Pareto efficient on the range) since the environment is partially knit. Thus, full efficiency is satisfied in this example because the range of the mechanism is the set of alternatives.

$\theta_3 = (m,i)$	$\theta_2 = (m,i)$	$\theta_2 = (m,g)$	$\theta_2 = (u,i)$	$\theta_2 = (u,g)$
$\theta_1 = (m,i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (m,g)$	AAA A	CCC C	AAA A	CCC C
$\theta_1 = (u,i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (u,g)$	AAA A	CCC C	AAA A	CCC C

$\theta_3 = (u,i)$	$\theta_2 = (m,i)$	$\theta_2 = (m,g)$	$\theta_2 = (u,i)$	$\theta_2 = (u,g)$
$\theta_1 = (m,i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (m,g)$	AAA A	CCA A/C	AAA A	CCA A/C
$\theta_1 = (u,i)$	AAA A	AAA A	AAA A	AAA A
$\theta_1 = (u,g)$	AAA A	CCA A/C	AAA A	CCA A/C

$\theta_3 = (m,g)$	$\theta_2 = (m,i)$	$\theta_2 = (m,g)$	$\theta_2 = (u,i)$	$\theta_2 = (u,g)$
$\theta_1 = (m,i)$	AAA A	CCC C	AAA A	CCC C
$\theta_1 = (m,g)$	CCC C	CCC C	CAC A/C	CCC C
$\theta_1 = (u,i)$	AAA A	ACC A/C	AAA A	ACC A/C
$\theta_1 = (u,g)$	CCC C	CCC C	CAC A/C	CCC C

$\theta_3 = (u,g))$	$\theta_2 = (m,i)$	$\theta_2 = (m,g)$	$\theta_2 = (u,i)$	$\theta_2 = (u,g)$
$\theta_1 = (m,i)$	AAA A	CCC C	AAA A	CCC C
$\theta_1 = (m,g)$	CCC C	CCC C	CAC A/C	CCC C
$\theta_1 = (u,i)$	AAA A	ACC A/C	AAA A	ACC A/C
$\theta_1 = (u,g)$	CCC C	CCC C	CAC A/C	CCC C

Table 5. Each agent's best alternative and outcomes of all voting by quota mechanisms.

4.2 Private goods without money

Example 4. Let $N = \{1, 2\}$ be a set of agents, $O = \{a, c\}$ be a set of objects. Each agent must be assigned one and only one object. Thus, the set of alternatives is $A = \{x = (a, c), z = (c, a)\}$, where the first component refers to the object that agent 1 gets. There is no money in this economy.

The type $\theta_i \in \Theta_i$ of each agent i is given by a signal s_i in $\Theta_i = [0, 1]$. Each individual $i \in N$ is endowed with a given auxiliary function $g_i : \Theta \to \mathbb{R}$ *increasing* in both signals.[19]

[19] Che, Kim, and Kojima (2015) also impose the following property which they call the *single-crossing property*: $\frac{\partial u_i(\theta)}{\partial s_i} > \frac{\partial u_j(\theta)}{\partial s_i}$ for any $\theta \in \Theta$. However, as they already mention, this condition is not required for the impossibility result to hold.

The preference function R is such that for each agent $i \in N$ and for each type profile $\theta \in \Theta = [0,1] \times [0,1]$, $R_i(\theta)$ is as follows: x is at least as good as z if and only if $g_i(\theta) \geq 0$.

The environment in Example 4 is knit (see Proposition 7 in Appendix B). Therefore by Theorem 1 only constant mechanisms can be ex post incentive compatible and respectful in this context.

Example 5. We consider the framework of Example 4, except that we change agents' preference functions to be induced by $g_1(s) = \min\left(median\left\{\frac{1}{4}, s_1, s_1, s_2\right\}\right) - \frac{1}{4}$ and $g_2(s) = \min\left(median\left\{\frac{1}{4}, s_2, s_2, s_1\right\}\right) - \frac{1}{4}$, respectively. That is, for each agent $i \in N$ and for each type profile $\theta \in \Theta$, $R_i(\theta)$ is as follows: x is at least as good as z if and only if $g_i(s) \geq 0$.

The main but significant difference between this example and the preceding one is that now the functions g_i are just weakly increasing.

Like in Example 3 above, the environment in this example is partially knit (see Proposition 8 in Appendix B) but not knit. To prove it, we consider the veto mechanisms defined below. Before introducing them we need the following definition: consider a partition of the signal (type) space and a useful graphical representation of it which is similar to the one defined in Che, Kim, and Kojima (2015).

Let $\{S_{ac}, S_{ca}, S_{aa}, S_{cc}, S^0\}$ be the partition of Θ where:
S^0 is the set of signal profiles for which both agents are indifferent between a and c,
S_{ac} is the set of signal profiles for which agent 1 prefers a to c, agent 2 prefers c to a, and the preferences are strict for at least one agent,
S_{ca} is equally defined after changing the roles of c and a,
S_{aa} is the set of signal profiles for which both agents prefer a to c, and
S_{cc} is equally defined after changing the roles of c and a.

In terms of alternatives, when the signals are in S_{ac} both agents prefer x to z, when they are in S_{ca} both prefer z to x, in S_{aa}, 1 prefers x over z and 2 prefers z over x, in S_{cc}, 1 prefers z over x and 2 prefers x over z, and in S^0 both are indifferent between x and z.

Now we say that a mechanism $f_{veto\ x}$ is a *veto rule for* x if for any type profile the outcome is agent 1's best alternative when it is unique, and it is agent 2's best alternative otherwise. Formally, for $\theta \in \Theta = [0,1] \times [0,1]$,

$$f_{veto\ x}(\theta) = \left\{ \begin{array}{c} x = (a,c) \text{ if } \theta \in S_{ca}, \text{ and} \\ z = (c,a) \text{ if } \theta \in S_{aa} \cup S_{ac} \cup S_{cc} \cup S^0 \end{array} \right\}.$$

In view of Theorem 1 the existence of these non-constant, ex post incentive compatible, and respectful mechanisms implies that the environment is no longer knit (in Lemma 1, Appendix B we show that veto rules satisfy the three properties). Now, Theorem 2 will ensure that these and other mechanisms that we may know to be ex post incentive compatible for our example will also be ex post group incentive compatible (therefore, Pareto efficient on the range) since the environment is partially knit. Thus, full efficiency is obtained in this example since the range is the whole set of alternatives.

4.3 Auctions

There is one unit of an indivisible good to be auctioned. Let N be the set of buyers (agents). An alternative in this model tells us which single agent, if any, gets the good and what positive price she pays for it, meaning then that the rest of agents do not get the good and pay zero. If no agent gets the good, no one pays anything. Formally, an alternative x is written as $x = (x_1, ...x_n) \in A = (\{0, 1\} \times \mathbb{R}_+)^n$, with $x_i = (a_i, p_i)$ where $a_i = 1$ and $p_i > 0$ if and only if agent i gets the good, and $p_l = 0$ for all agents l that do not get it.

We assume that agents' preferences are selfish. Agents only care about whether or not they are awarded the good and, if so, about how much they must pay for it. Therefore, we can define their preferences on the part of the alternative that concerns them and then naturally extend such preferences to alternatives.

The type θ_i of each agent i is given by a signal, $s_i \in \Theta_i \subseteq \mathbb{R}$ (where Θ_i has a minimum). Each individual $i \in N$ is endowed with a given auxiliary function $g_i : \Theta \rightarrow \mathbb{R}$ *non-decreasing* in her own signal s_i. The preference function R is such that for each agent $i \in N$ and for each type profile $\theta \in \Theta$, $R_i(\theta)$ is as follows:

(1) $(1, p_i)P_i(s)(1, q_i)$ for all $q_i > p_i$ (agent i strictly prefers paying less than more), and

(2) $(1, g_i(s))I_i(s)(0, 0)$ (agent i is indifferent between not getting the good and paying nothing or receiving the good and paying $g_i(s)$).

Notice that $g_i(s)$ is buyer i's valuation of the good, g_i has a minimum in Θ_i, and that the preference relation of i is fully determined once we know which alternative $(1, g_i(s))$ is indifferent to $(0, 0)$.

We assume all along this section that for each agent i, g_i satisfies the following standard condition in the literature: (a) g_i is non-decreasing in s_i.

Example 6. Let us assume that, in addition to condition (a), for any agent i, the evaluation will be the lowest possible if all other agents but i receive the lowest signal. This is formally expressed by condition: (b) $g_i(s) = g_i(\underline{s})$ for s such that $s_j = \underline{s}_j$ for all $j \in N\backslash\{i\}$.[20]

Under conditions (a) and (b), the environment in this example is knit[21] (see Proposition 9 in Appendix B). Hence, again by Theorem 1 we know that it will be impossible to design non-constant, ex post incentive compatible and respectful mechanisms in such framework. This negative result parallels those in Examples 2 and 5 above, where Theorem 1 also applies.

One could wonder whether it would be possible to find non-constant mechanisms by dropping the requirement of respectfulness. We do not have a full answer to this question, but the answer is negative if we substitute condition (a) by the stronger condition (c) g_i is

[20] An example of a g_i function satisfying these properties is presented by Jehiel, Meyer-Ter-Vehn, Moldovanu, and Zame (2006). In our notation, consider the case where $g_i(s) = \beta_i + \alpha \prod_{j \in N} s_j$, $\beta_i \in [0, 1]$, $\alpha > 0$ and the signal space is $S_i = [0, 1]$. Note that by fixing β_i and α, we have a unique preference formation rule for each agent.

[21] Our examples are chosen to illustrate our points, and the readers may want to create additional ones or to use them for comparison with alternative results. Take, for instance, the function $g_i(s) = \max\{s_1, ..., s_n\}$, that is used in Ivanov, Levin, and Niederle (2010), for other purposes. Such auxiliary function g_i satisfies condition (a) but not (b), and it could be used to define a knit (hence, also partially knit) environment. Since our purpose is only to provide some examples, we leave the possibility of constructing new ones based on this g_i to the interested readers.

strictly increasing in s_j for all $j \in N$, and the requirement that the good is always allocated. (See Proposition 10 in Appendix B.)

Now, Example 7 and our subsequent remarks will explore the positive consequences of apparently small changes in the preference function.

Example 7. For simplicity, let $N = \{1, 2\}$, $\Theta_i = \{0, 1\}$ for all $i \in N$ and $l, m, h \in \mathbb{R}_+$ with $0 = l < m < h$. The agent's preference function is defined as in the general framework but will now be based on a different auxiliary function that takes three possible values, low, medium and high.

More formally,

$$g_i(s) = \begin{cases} l & \text{if} \quad s_i = 0 \\ m & \text{if} \quad s_i = 1 \text{ and } s_j = 1, \\ h & \text{if} \quad s_i = 1 \text{ and } s_j = 0. \end{cases}$$

Observe that for each agent i, g_i satisfies (a) and the following condition:
(d) g_i is non-increasing in s_j, for all $j \in N \backslash \{i\}$.

Condition (d), in contrast to the cases encompassed in Proposition 10 and to some cases in Example 6, establishes that the valuation of the good by agent i depends negatively on other agents' signals. Note also that the function g_i in Example 7 does not satisfy condition (c).

Now, we assert that the environment in this example is not knit, but is partially knit (see Proposition 11 in Appendix B). Therefore, we can apply Theorem 2 and conclude that any ex post incentive compatible and respectful mechanism on that environment will also be ex post group incentive compatible, and therefore, Pareto efficient on the range.

In view of Theorem 1, to prove that is not knit, it is enough to show that the environment admits a non-constant, ex post incentive compatible, and respectful mechanism. Here is such a mechanism.[22] Let $l < p < m$ and $l < p' < m$. Let $f_{p,p'}$ be such that no agent gets the good if both signals are 0, agent 1 gets the good and pays p if her signal is 1, and agent 2 gets the good and pays p', otherwise. Formally, for $\theta \in \Theta = \{0, 1\} \times \{0, 1\}$,

$$f_{p,p'}(\theta) = \left\{ \begin{array}{c} ((0,0),(0,0)) \text{ if } s_1 = s_2 = 0, \\ ((1,p),(0,0)) \text{ if } s_1 = 1, \text{ and} \\ ((0,0),(1,p')) \text{ if } s_1 = 0,\ s_2 = 1. \end{array} \right\}$$

Let us complete the discussion of this and related examples with some additional comments. Example 7 provides a scenario where to apply Theorem 2, which is based on the assumption that changes in some agent's signal induce reverse effects in the preferences of the different participants in the auction. While we can think of environments and signals where this can be the case, the assumption that prevails in the literature on auctions is that all agents respond in the same direction to changes in some agent's signal. Led by this observation, we offer the reader the following additional remark (that is formally justified in Appendix B).

[22] In Lemma 4 in Appendix B we show that $f_{p,p'}$ is ex post incentive compatible and respectful defined on the environment (Θ, R) in Example 7.

Remark 2 *If we just modify Example 7 and assume that all agents' preferences respond in the same way positively to changes in signals, we can prove the existence in such setting of a mechanism that is respectful, and individually but not group ex post incentive compatible. Hence, this new specification leads to environments that are not partially knit.*

5 Discussion

In this paper we have emphasized the crucial role of environments, that is, combinations of a set of types and a preference function, in determining whether or not satisfactory ex post incentive compatible mechanisms can be designed.

Our classification of environments is not based on specific assumptions about preferences, or the structure of the space of alternatives, or other considerations that end up determining what combinations of types are admissible in specific applications. Rather, we have extracted from different possible special cases what we think are crucial aspects that distinguish some environments from others. These characteristics refer to how different type profiles are interconnected within a given set by means of the preference function.

We model the preferences of agents as binary relations, and conduct our analysis in ordinal terms.

Our conditions do not refer specifically to the structure of the set of types, or to its dimensionality. Since the distinction between one-dimensional and multidimensional signals is often seen as being determinant for the possibility or impossibility of designing efficient mechanisms with good incentive properties, our results suggest that this criterion, however important, needs not always be determinant.

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6 Appendix A

In this appendix we prove propositions stated in Section 2.

Proof of Proposition 1. Let $i \in N$ and $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $\theta_i \neq \widetilde{\theta}_i$ be such that $R_i(\theta_i) \neq R_i(\widetilde{\theta}_i)$. That is, $R_i(\theta_i, \theta_{N\backslash\{i\}}) \neq R_i(\widetilde{\theta}_i, \theta_{N\backslash\{i\}})$ for all $\theta_{N\backslash\{i\}} \in \times_{j\in N\backslash\{i\}}\Theta_j$ since (Θ, R) is a private values environment. Then, there will be a pair of alternatives, say x and z, such that $xP_i(\theta_i)z$ and $zR_i(\widetilde{\theta}_i)x$ (otherwise, for $\theta_i, \widetilde{\theta}_i \in \Theta_i$, $R_i(\theta_i) = R_i(\widetilde{\theta}_i)$). To show that the set of types Θ is not knit, we prove that for the two pairs $(x, (\theta_i, \theta_{N\backslash\{i\}}))$, $(z, (\widetilde{\theta}_i, \theta_{N\backslash\{i\}}))$, and whatever $\theta_{N\backslash\{i\}}$, there does not exist any θ', S, and $\widetilde{S}$ such that the passage from θ to θ' through S be x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ be z-satisfactory. We prove it by contradiction. Suppose otherwise that there exist θ^*, S^*, $\widetilde{S}^*$, such that the passages $\left\{m^h(\theta, S^*)\right\}_{h=0}^{t_{S^*}}$ and $\left\{m^h(\theta, \widetilde{S}^*)\right\}_{h=0}^{t_{\widetilde{S}^*}}$ from θ to θ^* through S^* and $\widetilde{\theta}$ to θ^* through $\widetilde{S}^*$ are x and z-satisfactory, respectively.
Since we are in a private values environment, changes in the type of agent j never affect the induced preferences of other agents, in particular never affect i's induced preferences if $j \neq i$. Moreover, we know that $xP_i(\theta_i, \theta_{N\backslash\{i\}})z$ and $zR_i(\widetilde{\theta}_i, \theta_{N\backslash\{i\}})x$. These two observations imply that agent i must belong to $I(S^*) \cup I(\widetilde{S}^*)$. That is, i will appear in at least one of these two sequences.
We concentrate on the steps of the passage where agent i changes her type and we show that there is no θ^* compatible with x-satisfactory and z-satisfactory passages from θ to θ^* and from $\widetilde{\theta}$ to θ^*.
Without loss of generality, by the remark just after Definition 4, we can assume that all types of agent i in S^* and $\widetilde{S}^*$ appear in the first positions in these sequences. Let's define $I_{S^*,i} \equiv \{h \in \{1, 2, ..., i_{S^*}\} : i(S^*, h) = i\}$ and $I_{\widetilde{S}^*,i} = \left\{h \in \{1, 2, ..., i_{\widetilde{S}^*}\} : i(\widetilde{S}^*, h) = i\right\}$.
Take $1 \in I_{S^*,i}$. Since $R_i^1(\theta, S^*)$ is an x-monotonic transform of $R_i(\theta_i, \theta_{N\backslash\{i\}})$, we have that $xP_i(m_i^1(\theta, S^*))z$. By repeating the same argument for each $h \in I_{S^*,i}$ we finally obtain that $xP_i(m_i^{i_{S^*}}(\theta, S^*))z$ where $m_i^{i_{S^*}}(\theta, S^*) = \theta_i^*$.
Now, take $1 \in I_{\widetilde{S}^*,i}$. Since $R_i^1(\widetilde{\theta}^*, \widetilde{S}^*)$ is a z-monotonic transform of $R_i(\widetilde{\theta}_i^*, , \theta_{N\backslash\{i\}})$, we have that $zR_i(m_i^1(\widetilde{\theta}^*, \widetilde{S}^*))x$. By repeating the same argument for each $h \in I_{\widetilde{S}^*,i}$ we finally obtain that $zR_i(m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*))z$ where $m_i^{i_{\widetilde{S}^*}}(\theta, \widetilde{S}^*) = \theta_i^*$.
As mentioned above, changes in types of agents different from i will not change agent i's preferences. Thus, we have obtained the desired contradiction. On the one hand that $xP_i(\theta^*)z$ and on the other hand, that $zR_i(\theta^*)x$. ■

Proof of Proposition 2. Two relevant observations: Remember that types are preferences, in that case, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in j's preferences do not affect i's preferences if $i \neq j$.
Let $\mathcal{U}$ denote the universal set of strict preferences in the classical social choice problem. Thus, $\mathcal{R}_i = \mathcal{U}$. To check for partial knitness , take any (x, R), $(z, \widetilde{R}) \in A \times \mathcal{U}^n$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N\backslash C(R, z, x)$. Without loss of generality, let $C(R, z, x) = \{1, 2, ..., c\}$ where c denotes its cardinality. Construct

S, $\widetilde{S}$ and R' satisfying the condition in partially knitness.
We shall denote, for each $R_i \in \mathcal{U}$, let R_i^z be the preferences obtained by lifting z to the first position and keep the relative position of all other alternatives.
Now, start from R and define $S = \{R_1^z, R_2^z, ..., R_c^z\}$ where $t_S = c$. Note that for each $h \in \{1, ..., c\}$, $R^h(R,S) \in \mathcal{U}^n$ and $R_i^h(R,S) = R_i^z \in \mathcal{U}$ is an x-reshuffling of i's previous preferences R_i. Then, $R' = R^c(R,S) = R^z \in \mathcal{U}^n$.
Now, start from $\widetilde{R}$ and define $\widetilde{S} = \left\{\widetilde{R}_1^z, \widetilde{R}_2^z, ..., \widetilde{R}_c^z, R_1^z, R_2^z, ..., R_c^z\right\}$ where $t_{\widetilde{S}} = 2c$. For each $h \in \{1, ..., c\}$, $R_i^h(\widetilde{R}, \widetilde{S}) = \widetilde{R}_i^z$ is a z-monotonic transform or a z-reshuffling (if z was already the top) of i's previous preferences $\widetilde{R}_i$, and for $h \in \{c+1, ..., 2c\}$, $R_i^h(\widetilde{R}, \widetilde{S}) = R_i^z \in \mathcal{U}$ is a z-reshuffling of i's previous preferences $\widetilde{R}_i^z$ Then, $R' = R^{2c}(\widetilde{R}, \widetilde{S}) = R^z$. ■

Proof of Proposition 3. The same two observations as in the proof of Proposition 2 apply: types are preferences, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in j's preferences do not affect i's preferences if $i \neq j$.
Let A be a finite and ordered set of alternatives in $\mathbb{R}$, the real line. For all $i \in N$, let $\mathcal{R}_i = \mathcal{S}$ be the set of strict single-peaked preferences on A according to the established real numbers order. We introduce some notation: Given $R_j \in \mathcal{S}$, $p(R_j)$ denotes the peak, that is, the best alternative, of R_j in A. Let $\overline{L}(R_i, x) = \{y \in A : xP_iy\}$ be the *strict lower contour set of* R_i *at* x. Given $R_j \in \mathcal{S}$ and $x \in A$, define $r(R_j, x)$ as the first alternative in $\overline{L}(R_j, x)$ in the opposite side of alternative x with respect to $p(R_j)$.
To check for partial knitness, take any (x, R), $(z, \widetilde{R}) \in A \times \mathcal{S}^n$ such that $\overline{C}(R, z, x) = C(R, z, x) \neq \varnothing$, $\#C(R, z, x) \geq 2$, and $\widetilde{R}_j = R_j$ for all $j \in N \backslash C(R, z, x)$. Without loss of generality, let $x < z$, which implies that $p(R_j) > x$. Also without loss of generality, let $C(R, z, x) = \{1, 2, ..., c\}$ where c denotes its cardinality. Now define $S = \widetilde{S} = C(R, z, x) = \{1, 2, ..., c\}$ and construct for each agent $j \in \{1, 2, ..., c\}$, R_j' depending on the cases below.
Take any $j \in C(R, z, x)$ and consider the following cases.
<u>Case 1</u>. $\widetilde{R}_j$ is such that $x\widetilde{P}_jz$. Take $R_j' \in \mathcal{S}$ such that $p(R_j') \in [x, z)$, $r(R_j, x) = z$, and $zP_j'y$ for all $y < x$. Notice that such R_j' exists, and the two following set inclusions hold: $\overline{L}(R_j, x) \subseteq \overline{L}(R_j', x)$, $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R_j', z)$. Thus, R_j' is both an x-monotonic transform *of* R_j and a z-monotonic transform of $\widetilde{R}_j$ (observe that with strict preferences, the above inclusion of strict lower contour sets is equivalent to Definition 2).
<u>Case 2</u>. $\widetilde{R}_j$ is such that $z\widetilde{P}_jx$. Consider several subcases.
<u>Case 2.1</u>. $\overline{L}(R_j, x) \subseteq \overline{L}(\widetilde{R}_j, x)$. Let $R_j' = \widetilde{R}_j$ and observe that R_j' is an x-monotonic transform of R_j (obviously, R_j' is a z-monotonic transform of $\widetilde{R}_j$ since $R_j' = \widetilde{R}_j$).
<u>Case 2.2</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$. We distinguish additional subcases which require different definitions of R_j'.
<u>*Case 2.2.1*</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R_j, z)$. Let $R_j' = R_j$ and observe that R_j' is an x-monotonic transform of R_j (obviously since $R_j' = R_j$) and R_j' is also a z-monotonic transform of $\widetilde{R}_j$.
<u>*Case 2.2.2*</u>. $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ *and* $\overline{L}(R_j, z) \subsetneqq \overline{L}(\widetilde{R}_j, z)$. This implies that either *(a)* $p(R_j), p(\widetilde{R}_j) \in (x, z)$ or else *(b)* $p(R_j), p(\widetilde{R}_j) > z$.

If *(a)* holds, then let R'_j be such that $p(R'_j) \in \Big[\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\Big]$, $r(R'_j, x) = r(R_j, x)$ and $r(R'_j, z) \geq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences R'_j exists.
If *(b)* holds, then let R'_j be such that $p(R'_j) \in \Big[z, \min\{p(R_j), p(\widetilde{R}_j)\}\Big]$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences R'_j exists.
Then, observe that R'_j defined in *(a)* and *(b)* is both an x-monotonic transform of R_j and a z-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.
<u>*Case 2.2.3*</u>: $\overline{L}(\widetilde{R}_j, x) \subsetneqq \overline{L}(R_j, x)$ and $z \in \Big(\min\{p(R_j), p(\widetilde{R}_j)\}, \max\{p(R_j), p(\widetilde{R}_j)\}\Big)$. Assume that $p(R_j) < z < p(\widetilde{R}_j)$, otherwise, a similar argument would work.
This implies that either *(a)* $r(R_j, x) \in \Big(z, p(\widetilde{R}_j)\Big]$ or *(b)* $r(R_j, x) \in \Big(p(\widetilde{R}_j), r(\widetilde{R}_j, x)\Big)$ holds.
If *(a)* holds, then let R'_j be such that $p(R'_j) \in [z, r(R_j, x))$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$. By definition of single-peakedness, such preferences R'_j exists.
If *(b)* holds, then let R'_j be such that $p(R'_j) \in \Big[z, \min\{r(R_j, x), r(\widetilde{R}_j, z)\}\Big)$, $r(R'_j, x) \leq r(R_j, x)$ and $r(R'_j, z) \leq r(\widetilde{R}_j, z)$.
Then, observe that R'_j in *(a)* and *(b)* is both an x-monotonic transform of R_j and a z-monotonic transform of $\widetilde{R}_j$ since $\overline{L}(R_j, x) \subseteq \overline{L}(R'_j, x)$ and $\overline{L}(\widetilde{R}_j, z) \subseteq \overline{L}(R'_j, z)$ hold.
Finally, for each $j \in C(R, z, x)$ we repeat the same argument. ■

Proof of Proposition 4. The same two observations as in the proof of Propositions 2 and 3 apply: types are preferences, that is, $\theta_i = R_i \in \mathcal{R}_i = \Theta_i$ for each $i \in N$. Moreover, changes in j's preferences do not affect i's preferences if $i \neq j$.
The proof follows the same argument as the one in Proposition 2, given that agents have all possible strict preferences over individual assignments and preferences are selfish. As in Barberà, Berga, and Moreno (2016), just note that although preferences over individual assignments are strict, preferences over alternatives allow for indifferences, by selfishness: all alternatives with the same individual assignment are indifferent for such individual agent. Thus, in the case of housing $C(R, z, x) \supseteq \overline{C}(R, z, x)$ holds and R_i^z are the preferences obtained by lifting z and also all alternatives with the same individual assignment z_i to the first position and keep the relative position of all other alternatives. ■

7 Appendix B

In this appendix we present some aspects of the applications in Section 4 with more detail, in order to prove knitness or partially knitness of the environments defined in Examples 2 to 7. We also state and prove some intermediate results required for the auctions application.

Deliberative juries

Example 2 (continued)

Proposition 5 *The environment* (Θ, R) *in Example 2 is knit.*

Proof. To prove knitness we just need to combine the following two results.
(1) Consider a pair formed by (A, θ) for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. Let $\theta' \in \Theta$ be such that $\theta'_1 = (l, i)$ and $\theta'_j = (h, i)$ for any $j \in N \backslash \{1\}$. We now define the sequence S to sequentially go from type profile θ to type profile θ' by successively changing the type of the agents in S while preserving A-satisfactoriness. First change, one by one and in any order, agents' signals from $s_j \neq i$ to i. By definition of l and h, in each of the above changes, the induced preferences of the agent changing her type is an A-monotonic transform of her previous preferences (sometimes an A-reshuffling).
Observe that by definition of the preference functions, the following condition is satisfied: if $\widehat{s}_j = i$ for all $j \in N$, all jurors prefer A to C for any $\widehat{b}_j \in B$.
We now change, one by one and in any order, each agent's $b_j \neq h$ from b_j to h for any $j \in N \backslash \{1\}$ and from $b_1 \neq l$ to l in the case of agent 1. By the observation made just above, in each of these changes, the induced preferences of each agent is the same and therefore they are an A-reshuffling of their previous preferences. Then, we have defined S such that θ leads to θ' through S and the passage from θ to θ' is A-satisfactory.
(2) Consider a pair (C, θ) for any $\theta \in \Theta$ where $\theta_j = (b_j, s_j)$ for each $j \in N$. We now define the sequence S to go from type profile θ to θ' above by successively changing the type of the agents in S while preserving C-satisfactoriness. First change, one by one and in any order, agents from $s_j \neq g$ to g. By definition of l and h, in each of the above changes, the induced preferences of the agent changing her type is a C-monotonic transform of her previous preferences (sometimes a C-reshuffling).
Observe that by definition of the preference function, the following property is satisfied: if $\widehat{s}_j = g$ for all $j \in N$, all jurors prefer C to A for any $\widehat{b}_j \in B$.
We now change one by one, and in any order, each agent's $b_j \neq h$ from b_j to h for any $j \in N \backslash \{1\}$ and from $b_1 \neq l$ to l in the case of agent 1. By the observation made just above, in each of these steps, the preferences of the agents stay the same and therefore they are a C-reshuffling of their previous ones. After that, we change the signal of the agent 1 from g to i. This implies that the preferences of agent 1 remain identical, but those of all others go from C preferred to A, to A preferred to C, given that $b_j = h$ for any $j \in N \backslash \{1\}$. Finally, we change the type of the rest of the agents one by one from g to i. In each one of these steps the preferences of the agent that moves is still A preferred to C. The passage from θ to θ' is C-satisfactory by construction. ■

Example 3 (continued)

Before engaging in the proof that the environment in Example 3 is partially knit (see Proposition 6), we develop the argument for a particular example.

Consider a particular pair of types and alternatives, (A, θ) and $(C, \widetilde{\theta})$ where $\theta = ((u, g), (u, i), (m, g))$ and $\widetilde{\theta} = ((m, i), (u, i), (u, g))$. Let $\theta' = ((m, i), (u, i), (m, g))$. The profiles of preferences they induce are shown in Table 6.

$R(\theta) = R((u, g), (u, i), (m, g))$	$R(\widetilde{\theta}) = R((m, i), (u, i), (u, g))$	$R(\theta') = R((m, i), (u, i), (m, g))$
C A C	A A A	A A A
A C A	C C C	C C C

Table 6: Aents' preferences induced by θ, $\widetilde{\theta}$, and θ', respectively.

We can check that $\overline{C}(\theta, C, A) = C(\theta, C, A) = \{1, 3\}$ and $\widetilde{\theta}_2 = \theta_2$ (that is, requirements in Definition 6 are satisfied). As shown in Table 7 below, it is possible to move from θ to θ' by successively changing, one by one, the type of the agents. In this case, agent 1 from (u, g) to (m, i). According to our notation, $I(S) = \{1\}$. Likewise, as shown in Table 8 below, we can move from $\widetilde{\theta}$ to θ' by successively changing, one by one, the type of some agents. In this case, agent 3 from (u, g) to (m, g), that is, $I(\widetilde{S}) = \{3\}$. In Table 7, note that the preferences $R_1(\theta')$ of agent 1 are an A-monotonic transform of her previous ones, which also involve a change of those for agent 3. Similarly, notice that the preferences $R_3(\theta')$ of 3 in Table 8 are a C-reshuffling of her previous ones.

$R(\theta) = R, ((ug), (u, i), (m, g))$	$R(\theta') = R((\mathbf{m}, \mathbf{i}), (u, i), (m, g))$
C A C	A A A
A C A	C C C

Table 7: Induced agents' preferences given the specified type changes from θ to θ'.

$R(\widetilde{\theta}) = R((m, i), (u, i), (u, g))$	$R(\theta') = R((m, i), (u, i), (\mathbf{m}, g))$
A A A	A A A
C C C	C C C

Table 8: Induced agents' preferences given the specified type changes from $\widetilde{\theta}$ to θ'.

In Tables 7 and 8, we have illustrated the idea of partial knitness for two given type profiles. We now show that any relevant pair of type profiles are connected through two appropriate sequences.

Proposition 6 *The environment* (Θ, R) *in Example 3 is partially knit.*

Proof. Take two pairs (A, θ), $(C, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, C, A) = C(\theta, C, A) \neq \varnothing$, $\#C(\theta, C, A) \geq 2$, and for $j \in N \backslash C(\theta, C, A)$, $\widetilde{\theta}_j = \theta_j$. By definition, for all $j \in N$, $\theta_j = (b_j, s_j)$ and $\widetilde{\theta}_j = (\widetilde{b}_j, \widetilde{s}_j)$. We have to show that there exist $\theta' \in \Theta$ and sequences of types S and $\widetilde{S}$

such that θ leads to θ' through S, $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$, and the passages from θ and $\widetilde{\theta}$ to θ' are, respectively, A and C-satisfactory.

Let $\theta' \in \Theta$ be such that $\theta'_j = (b_j, g)$ for any $j \in C(\theta, C, A)$ and $\theta'_j = \theta_j$ for any $j \in N \backslash C(\theta, C, A)$. Define the sequence $S = \{(b_k, g)\}$, where $k \in C(\theta, C, A)$ and $s_k = i$. Note that $I(S)$ is either a singleton or empty. If the latter, let θ' be θ.

By definition of the preference function in the example, if some agent j prefers C to A, the signal profile must be such that at most one agent k has signal i: $s_k = i$. Thus, S is well-defined. Moreover, $b_k = m$ since for unswerving jurors to have C over A their signal must be g. And by definition of m increasing the support for g implies that preferences remain C over A for agent k (i.e. and A-reshuffling) and will be C over A for the other agents.

Therefore, we have defined S to go from θ to θ' through S and the passage is A-satisfactory. We now go from $\widetilde{\theta}$ to θ' by successively changing the type of the agents in $C(\theta, C, A)$, one by one in any order, from to $\widetilde{s}_j \neq g$ to g. This set of agents are those in $I(\widetilde{S})$.

By definition of the preference function, if one agent changes her signal by increasing the support for a guilty veredict, then each agents' induced preferences remain either the same as before or change in favor of C. Thus, in each one of the above changes, the induced preferences of the agent changing her type is a C-monotonic transform of her previous ones (sometimes a C-reshuffling).

Now, take any two pairs (C, θ), $(A, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, A, C) = C(\theta, A, C) \neq \varnothing$, $\#C(\theta, A, C) \geq 2$, and for $j \in N \backslash C(\theta, A, C)$, $\widetilde{\theta}_j = \theta_j$, a similar argument would work but defining $\theta' \in \Theta$ to be such that $\theta'_j = (b_j, i)$ for any $j \in C(\theta, A, C)$ and $\theta'_j = \theta_j$ for any $j \in N \backslash C(\theta, A, C)$. Define the sequence $S = \{(b_k, i)\}$, where $k \in C(\theta, A, C)$ and $s_k = g$. Note that $I(S)$ is either a singleton or empty. If the latter, let θ' be θ.

Again, by definition of the preference function in the example, if some agent j prefers A to C, the signal profile must be such that only one single agent, or at most two, have signal g. In the latter case, none of the two are agent j, and both have preferences C over A. Thus, S is well-defined. Moreover, by definition of m and u increasing if the single agent with signal g says i, that preferences of this agent and those of all other agents will be A over C.

Therefore, we have defined S to go from θ to θ' through S and the passage is A-satisfactory. We now sequentially go from $\widetilde{\theta}$ to θ' by successively changing the type of the agents in $C(\theta, A, C)$, one by one in any order, from to $\widetilde{s}_j \neq i$ to i. This set of agents are those in $I(\widetilde{S})$.

By definition of agents' preference function, if one agent changes her signal by increasing the support for veredict of innocence, then each agents' induced preferences remain either the same as before or change in favor of A. Thus, in each one of the above changes, the induced preferences of the agent changing her type is a A-monotonic transform of her previous ones (sometimes a A-reshuffling). ∎

Private goods without money

Example 4 (continued)

We shall prove that the environment in this example is knit. In Example 4, we assume that the sets S_{ac} and S_{ca} are non-empty.

Figure 1 provides a generic representation of these sets whose frontiers correspond to the pairs of signals leading to agents' indifference curves over alternatives: $\{\theta \in \Theta = [0,1] \times [0,1] : xI_i(\theta)y\}$. Since we have assumed that g_i is increasing in both signals, agents' indifference curves are strictly decreasing, and since S_{ac} and S_{ca} are non-empty the two curves will have an interior intersection.[23]

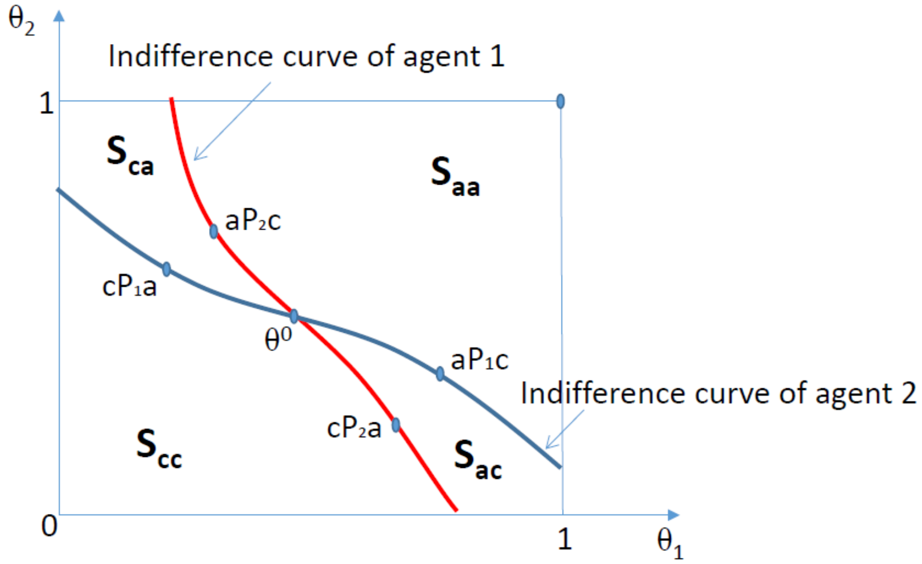


Figure 1. Examples of the partition of S in Example 4.

We can now state and prove Proposition 7.

Proposition 7 *The environment* (Θ, R) *in Example 4 is knit.*

Proof. Given any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ we will show that there exist θ', S, $\widetilde{S}$ such that θ leads to θ' through S, $\widetilde{\theta}$ leads to θ' through $\widetilde{S}$ and the passages are x and z-satisfactory. We choose $\theta' = (1, 1)$ independently of the two chosen pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. In defining the sequence S from θ to θ' with x as reference alternative, we distinguish two cases where we will end up analyzing all possible $\theta \in \Theta$. In particular, we cover the case where θ and $\widetilde{\theta}$ are the same.

Case 1. $\theta \in S_{ca} \cup S_{aa} \cup S^0$. First change the type of agent 1 from $\theta_1 \neq 1$ to 1. Since the function g_1 is increasing in type 1, the preferences of agent 1 induced by this change are either an x-reshuffling (if $\theta \in S_{aa}$) or an x-monotonic transform ($\theta \in S_{ca} \cup S^0$) of her original ones. Then change the type of agent 2 from θ_2 to 1. Again, since the function g_2 is increasing in type 2, the preferences of agent 2 induced by this change are an x-reshuffling of her original ones (see Picture 2.a in Figure 2).

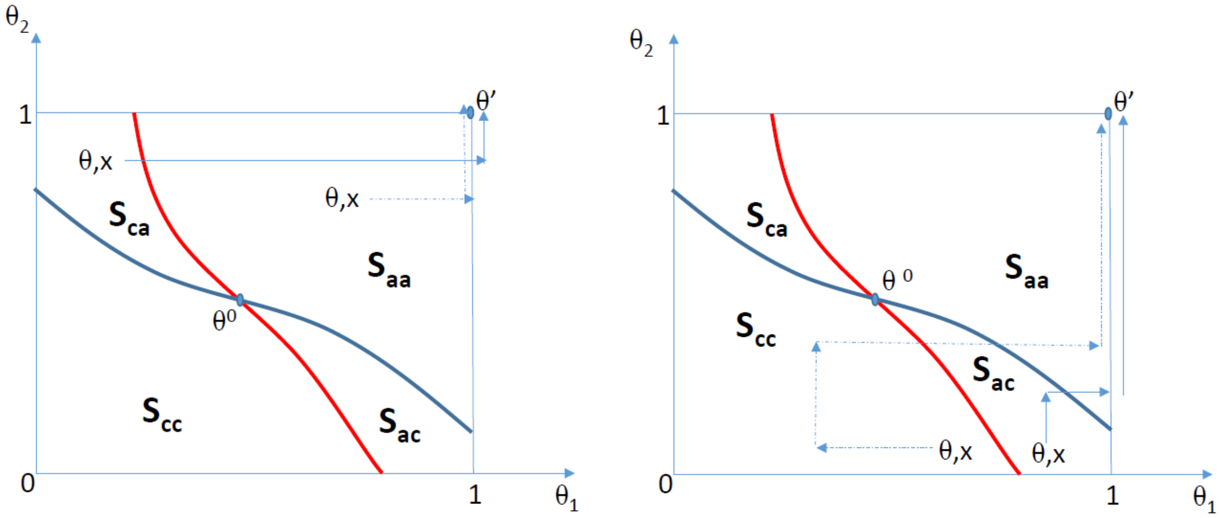
Case 2. $\theta \in S_{ac} \cup S_{cc}$. In this case we may not be able to change types of agents from $\theta_i \neq 1$ to $(1, 1)$ as directly as above.

If θ is a type profile from which we could reach another one in S_{aa} by letting the type of the first agent to be 1, we use the same argument as in Case 1: first change the type of agent 1 from $\theta_1 \neq 1$ to 1. The preferences of agent 1 induced by this change are either an x-reshuffling (if $\theta \in S_{ac}$) or an x-monotonic transform (if $\theta \in S_{cc}$) of her original ones. Then change the type of agent 2 from θ_2 to 1. The preferences of agent 2 induced by this change

[23] Although in all pictures corresponding to this example the indifference curves only intersect once, our formal arguments apply to the multiple intersection case.

are an x-reshuffling of her original ones.
If not, before reaching this situation, the sequence S must start by previous changes of signals, at most one for each agent, as shown in Picture 2.b in Figure 2, that keep us within the element of the partition where θ belongs to. The induced preferences resulting from these previous type changes remain unchanged.

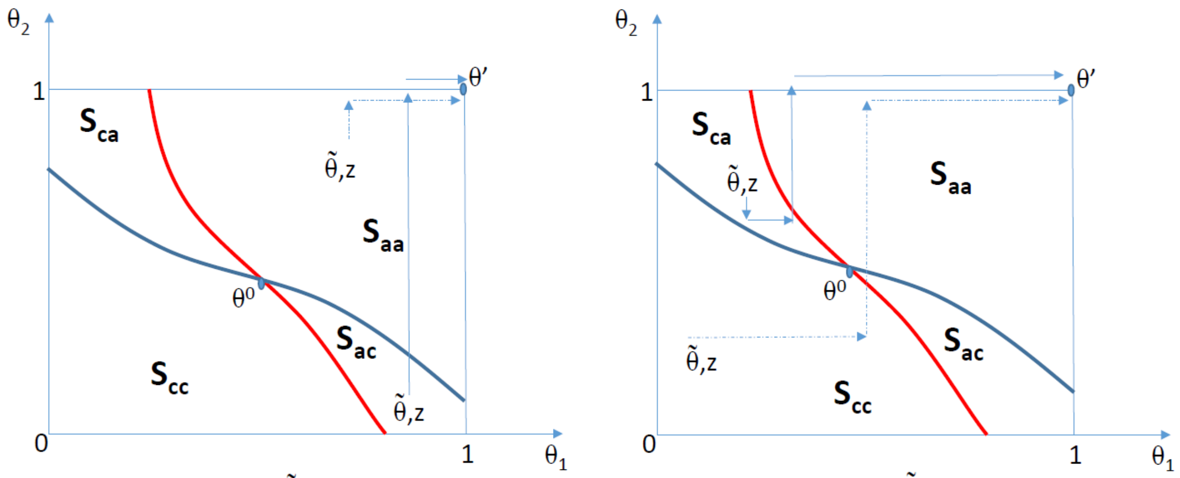
To define the sequence $\widetilde{S}$ from $\widetilde{\theta}$ to θ' with z as reference alternative, we would follow a parallel construction to Cases 1 and 2 above. The relevant cases would now be Case 3: $\widetilde{\theta} \in S_{ac} \cup S_{aa} \cup S^0$ and Case 4: $\widetilde{\theta} \in S_{ca} \cup S_{cc}$ where we would consider all possible type



Picture 2.a. Case 1: $\theta \in S_{ca} U S_{aa} U s^0$ Picture 2.b. Case 2: $\theta \in S_{ac} U S_{cc}$

Figure 2. Changes of agents' types in Cases 1 and 2, proof of Proposition 7.

profiles $\widetilde{\theta} \in \Theta$ including θ. The proof for the existence of the sequence $\widetilde{S}$ would require a similar argument to those of Cases 1 and 2, respectively, but changing first agent 2's signal to 1 when required to get to S_{aa}. See the graphical representation in Figure 3.
The construction of these passages proves that our environment is knit as we wanted to show.



Picture 3.a. Case 3: $\tilde{\theta} \in S_{ac} \cup S_{aa} \cup s^0$ Picture 3.b. Case 4: $\tilde{\theta} \in S_{ca} \cup S_{cc}$

Figure 3. Changes of agents' types in Cases 3 and 4, proof of Proposition 7.

∎

Example 5 (continued)

Before engaging in the proof that the environment in Example 5 is partially knit, observe that the changes in the functions g_i imply that the sets $\overline{S}_{ca} = \{\theta \in \Theta : zP_1x \text{ and } zP_2x\}$ and $\overline{S}_{ac} = \{\theta \in \Theta : xP_1z \text{ and } xP_2z\}$ are empty, and that S^0 is not a singleton. Due to the specific form of g_i the indifference set is L-shaped and thick, as shown in Figure 4.

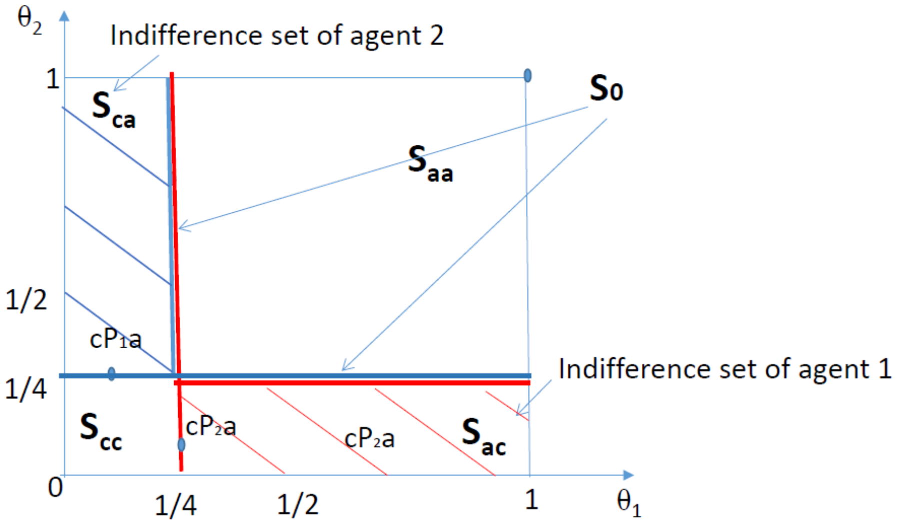


Figure 4. Partition of S in Example 5.

Proposition 8 *The environment* (Θ, R) *in Example 5 is partially knit.*

Proof. Remember that type profiles are signal profiles. Thus, we identify s with θ. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, z, x) \neq \varnothing$ and $\#C(\theta, z, x) \geq 2$. These two

conditions on θ imply that we must only consider $\theta \in S_{ca}$, i.e. where agent 1 strictly prefers z to x and agent 2 is indifferent between x and z. Define $\theta' = \widetilde{\theta}$.
We have to define S such that θ leads to $\theta' = \widetilde{\theta}$ through S and the passage is x-satisfactory. We distinguish two cases. See the graphical representation of both cases in Figure 5.
Case 1. $\widetilde{\theta} \in S_{aa} \cup S_{ca}$. Define $S = \{\widetilde{\theta}_1, \widetilde{\theta}_2\}$ and $I(S) = \{1, 2\}$. Note that if $\widetilde{\theta}, \theta \in S_{ca}$ the proof is obvious since we move along the same set S_{ca} and no agent preferences change. Suppose that $\widetilde{\theta} \in S_{aa}$. We first increase the signal of agent 1 to $\theta'_1 = \widetilde{\theta}_1$. The induced preferences of agent 1 are an x-monotonic transform of her previous ones. Agent 2 turns to strictly prefer z to x, that is, $zR_2(\theta'_1, \theta_2)x$. Decrease or increase now agent 2's signal to $\theta'_2 = \widetilde{\theta}_2$. Note that agent 2's induced preferences are identical to her previous ones, thus, are obviously an x-reshuffling of them. So we have gone from θ to θ' through adequate types changes with respect to x.
Case 2. $\widetilde{\theta} \in S_{cc} \cup S_{ac}$. Define $S = \{\widetilde{\theta}_2, \widetilde{\theta}_1\}$ and $I(S) = \{2, 1\}$. We first decrease the signal of agent 2 to $\theta'_2 = \widetilde{\theta}_2$. The induced preferences of agent 2 are an x-monotonic transform of her previous ones $R_2(\theta)$ (since $zP_2(\theta)x$ while $xP_2(\theta_1, \theta'_2)z$). Agent 1 turns to have the same preferences as before, that is, $zR_1(\theta_1, \theta'_2)x$. Now, we decrease or increase agent 1's signal to $\theta'_1 = \widetilde{\theta}_1$. Note that agent 2's induced preferences are either identical to her previous ones (thus, obviously an x-reshuffling of those) or an x-monotonic transform of $R_1(\theta_1, \theta'_2)$ (since $zP_1(\theta_1, \theta'_2)x$ while $zI_1(\theta'_1)x$). So, we have gone from θ to θ' through adequate changes of types with reference x.
It would remain to consider any two pairs where (z, θ), $(x, \widetilde{\theta}) \in A \times \Theta$ are such that $\overline{C}(\theta, x, z) \neq \varnothing$ and $\#C(\theta, x, z) \geq 2$, a symmetric and similar argument would work.

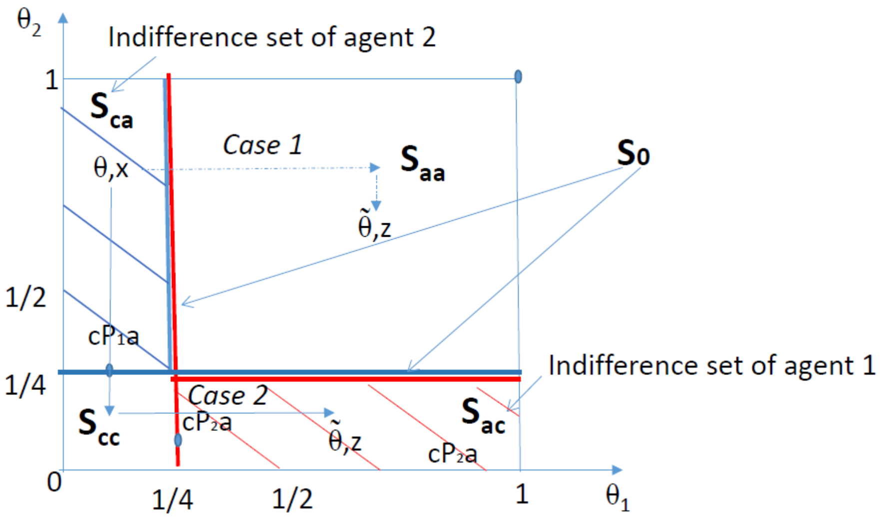


Figure 5. Changes of agents' types, proof of Proposition 8.

■

Finally, we show that the mechanism $f_{veto\ x}$ defined in Section 4.2 is non-constant, satisfies ex post incentive compatibility and respectfulness in the environment (Θ, R) defined in Example 5. which implies, by Theorem 1, that the environment in Example 5 is not knit.

Lemma 1 *$f_{veto,x}$ is non-constant, ex post incentive compatible, and respectful in the environment (Θ, R) in Example 5.*

Proof. Observe that, by definition, $f_{veto\ x}$ is non-constant and no agent can gain by changing her individual types, since she will either obtain the same or an indifferent one, when deviating, or else obtain her best outcome through by being truthful. Ex post group incentive compatibility is straightforward since changing both types it is impossible to weakly improve both agents, and at least on of them strictly: Note that either agent 1 or 2 strictly lose (we need to check 6 cases: $\theta \in S_{aa}$ and $\theta' \in S_{ca}$ or viceversa; $\theta \in S_{ac}$ and $\theta' \in S_{ca}$ or viceversa; and $\theta \in S_{cc}$ and $\theta' \in S_{ac}$ or viceversa). To show that $f_{veto\ x}$ is respectful, note that the only way for agent 1 to remain indifferent according to her initial preferences $R_1(\theta)$ and get a different outcome when changing her type is when $\theta \in S_{ac}$ and $\theta'_1 < \frac{1}{4}$ such that $(\theta'_1, \theta_2) \in S_{cc}$. However, $R_1(\theta'_1, \theta_2)$ is not an $x = f_{veto\ x}(\theta)$-monotonic transform of $R_1(\theta)$. Similarly, for agent 2, to remain indifferent and get a different outcome when changing her type $\theta \in S^0$ and $\theta_2 \geq \frac{1}{4}$, $\theta'_2 < \frac{1}{4}$. However, $R_2(\theta_1, \theta'_2)$ is not a $z = f_{veto\ x}(\theta)$-monotonic transform of $R_2(\theta)$. ■

Auctions

Example 6 (continued)

The following Lemma 2 is used in the proofs of Propositions 9 and 11 below.

Lemma 2 *Let g_k be non-decreasing in s_k. For all $s \in \Theta$, $R_k(s'_k, s_{-k})$ is a y-monotonic transform of $R_k(s)$ for all $s'_k < s_k$, $k \in N$ and $y \in A$ such that $y_k = (0, 0)$.*

Proof. Take $s \in \Theta$, $k \in N$ and $y \in A$ such that $y_k = (0, 0)$ and $s'_k < s_k$. Since g_k is non-decreasing in s_k, $g_k(s'_k, s_{N\setminus\{k\}}) \leq g_k(s)$ which means that agent k values the good in signal profile $(s'_k, s_{N\setminus\{k\}})$ at most as under profile s. Thus, $(0, 0)$ weakly improves its position in $R_k(s'_k, s_{N\setminus\{k\}})$ compared to its position in $R_k(s)$. Formally, $R_k(s'_k, s_{N\setminus\{k\}})$ is a y-monotonic transform of $R_k(s)$. ■

Proposition 9 *The environment (Θ, R) in Example 6 is knit.*

Proof. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$. We must find θ', sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Consider $\theta' = \left(\widetilde{s}_i, \underline{s}_{N\setminus\{i\}}\right)$. We first propose a sequence of types $S = \underline{s}$ $(t_S = n)$ with $I(S)$ defined as follows. We initially change, one by one, the signal of agents that do not get the good in x from s_k to $\underline{s}_k$ following the order of natural numbers. If there is one agent i left who was getting the good in x change her signal from s_i to $\underline{s}_i$. In each step $h \in \{1, ..., n-1\}$, by Lemma 2, we obtain that $R_h(m_h(\theta, S))$ is an x-monotonic transform of $R_h(m_{h-1}(\theta, S))$ since no agent h gets the good in x.

As for the last agent in the sequence, her preferences will not change when her signal goes from s_i to $\underline{s}_i$ due to assumption (b) of function g_i.

This completes our argument that the passage from θ to θ' through S is x-satisfactory.

We could repeat exactly the same argument to show that the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory after replacing the roles of θ by $\widetilde{\theta}$ and x by z. ■

Proposition 10 *Any ex post group incentive compatible mechanism that always allocates the good to some agent is constant, if all auxiliary functions g_i satisfy (b) and (c).*

Proof. Take $x = f(\overline{s})$ and without loss of generality suppose that agent 1 gets the good and pays p. We show that f is constant by the following steps.

Step 1. For any $s = (\overline{s}_1, s_2, ...s_n)$, such that $s_j \in \Theta_j$ for $j \in N \backslash \{1\}$, then agent 1 gets the good and pays p.

Take any agent that does not get the good, without loss of generality, say agent 2. Consider $s = (s_2, \overline{s}_{N \backslash \{2\}})$ where $s_2 < \overline{s}_2$. By condition (c) of g_i (that is, g_i is strictly increasing in all s_j, $j \in N$), $g_2(s_2, \overline{s}_{N\backslash\{2\}}) < g_2(\overline{s})$. Ex post incentive compatibility implies that $f_2(s_2, \overline{s}_{N\backslash\{2\}}) = f_2(\overline{s}) = (0,0)$. Take now any other agent $k \in N\backslash\{1,2\}$. By condition (c), for each $k \in N\backslash\{1,2\}$, $g_k(s_2, \overline{s}_{N\backslash\{2\}}) < g_k(\overline{s})$. Thus, by ex post group incentive compatibility, we get that for any $k \in N\backslash\{1,2\}$ $f_k(s_2, \overline{s}_{N\backslash\{2\}}) = f_k(\overline{s}) = (0,0)$. This implies that agent 1 gets the good at $(s_2, \overline{s}_{N\backslash\{2\}})$. Moreover, agent 1 pays the same price p. Otherwise, if $p' < p$, coalition N would profitably deviate from $\overline{s}$ to $(s_2, \overline{s}_{N\backslash\{2\}})$ since $f_1(s_2, \overline{s}_{N\backslash\{2\}}) = (1, p') P_1(\overline{s}) f_1(\overline{s}) = (1,p)$. The other way around if $p' > p$.

Repeating $n-2$ additional times the same argument, one for each agent $j \in N\backslash\{1,2\}$, we obtain that for any $s = (\overline{s}_1, s_2, ...s_n)$, such that $s_j \in \Theta_j$ for $j \in N\backslash\{1\}$, agent 1 gets the good and pays p.

Step 2. For any $s = (s_1, \underline{s}_{N\backslash\{1\}})$, such that $s_1 \in \Theta_1$, then agent 1 gets the good and pays p.

By condition (b) of g_1, $g_1(s_1, \underline{s}_{N\backslash\{1\}}) = g_1(\underline{s})$ for any $s_1 \in \Theta_1$. Thus, for any $s_1 \in \Theta_1$, agent 1's preferences $R_1(s_1, \underline{s}_{N\backslash\{1\}})$ coincide with $R_1(\underline{s})$. In particular, $R_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$ coincide with $R_1(\underline{s})$. By ex post incentive compatibility, for all $s_1 \in \Theta_1$, $f_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}}) I_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}}) f_1(s_1, \underline{s}_{N\backslash\{1\}})$, being $f_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}}) = (1,p)$ by Step 1. If agent 1 gets the good at $(s_1, \underline{s}_{N\backslash\{1\}})$, since $R_1(s_1, \underline{s}_{N\backslash\{1\}})$ coincide with $R_1(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$, the price must be the same. That is, $f_1(s_1, \underline{s}_{N\backslash\{1\}}) = (1,p)$ and then the proof of Step 2 ends. Otherwise, take $l \in N\backslash\{1\}$ such that $f_l(s_1, \underline{s}_{N\backslash\{1\}}) = (1, p_l)$. By condition (c) of g_l, $g_l(s_1, \underline{s}_{N\backslash\{1\}}) < g_l(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$. Note that if $p_l < g_l(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$, coalition $\{1, l\}$ can ex post profitably deviate at $(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$ via $(s_1, \underline{s}_l)$ (agent l would strictly gain while 1 remain indifferent). If $p_l \geq g_l(\overline{s}_1, \underline{s}_{N\backslash\{1\}})$, coalition $\{1,l\}$ can ex post profitably deviate at $(s_1, \underline{s}_{N\backslash\{1\}})$ via $(\overline{s}_1, \underline{s}_l)$ (agent l would strictly gain). Thus, we have shown that agent 1 gets the good and pays p at $(s_1, \underline{s}_{N\backslash\{1\}})$, for any $s_1 \in \Theta_1$.

Step 3. For any $s \in \Theta$ such that $s_1 < \overline{s}_1$ and there exists $l \in N\backslash\{1\}$ such that $s_l > \underline{s}_l$, then agent 1 gets the good and pays p.

Let $C = \{i \in N\backslash\{1\} : s_i > \underline{s}_i\}$.

First, observe that if agent 1 gets the good at s in Step 3, by ex post incentive compatibility, the price must be p. Otherwise, 1 could ex post profitably deviate at s via $\overline{s}_1$ if $p' > p$, or at $(\overline{s}_1, s_{N\backslash\{1\}})$ via s_1 if $p > p'$. Consider the following two cases for which we obtain a contradiction.

Step 3.1. Agent $k \in N\backslash\{1\}$ gets the good at s and $s_k > \underline{s}_k$.

Take an agent $j \in C\backslash\{k\}$ who does not get the good at s and change her type from s_j

to $\underline{s}_j$. If $C\backslash\{k\}$ is empty, we have that $f_k(s_k, \underline{s}_{N\backslash\{k\}}) = (1, p')$, and by applying the same argument as in Step 2 we would get $f_k(\underline{s}) = (1, p')$, which contradicts the reasoning in Step 2 when applied to $\underline{s}$. Otherwise, by condition (c) of g_j, $g_j(\underline{s}_j, s_{N\backslash\{j\}}) < g_j(s_j, s_{N\backslash\{j\}})$. By ex post incentive compatibility, $f_j(s) = f_j(\underline{s}_j, s_{N\backslash\{j\}}) = (0, 0)$. Again, if $C\backslash\{k, j\}$ is empty, we have that $f_k(s_k, \underline{s}_{N\backslash\{k\}}) = (1, p')$ and by applying the same argument as in Step 2 we would get $f_k(\underline{s}) = (1, p')$ which contradicts Step 2 applied to $\underline{s}$. Otherwise, take $j' \in C\backslash\{k, j\}$, and by condition (c) of $g_{j'}$, $g_{j'}(\underline{s}_j, s_{N\backslash\{j\}}) < g_{j'}(s_j, s_{N\backslash\{j\}})$. If for any $j' \in N\backslash\{k\}$, $f_{j'}(\underline{s}_j, s_{N\backslash\{j\}}) = f_{j'}(s_j, s_{N\backslash\{j\}}) = (0, 0)$, we obtain that $f_k(\underline{s}_j, s_{N\backslash\{j\}}) = (1, p')$ and we repeat the same argument in Step 3.1 for $l \in C\backslash\{k, j\}$. If for some j', $f_{j'}(\underline{s}_j, s_{N\backslash\{j\}}) \neq (0, 0)$, we would get a contradiction to ex post group incentive compatibility: $\{j, j'\}$ would profitably deviate from $(\underline{s}_j, s_{N\backslash\{j\}})$ via $(s_j, s_{j'})$ if $p' > g_j(\underline{s}_j, s_{N\backslash\{j\}})$ and from $(s_j, s_{N\backslash\{j\}})$ via $(\underline{s}_j, s_{j'})$ if $p' \leq g_j(\underline{s}_j, s_{N\backslash\{j\}})$. Thus, $f_k(\underline{s}_j, s_{N\backslash\{j\}}) = (1, p')$.
By repeating the same argument, and changing one by one the signal from s_l to $\underline{s}_l$ for each $l \in C\backslash\{k\}$, we obtain that $f_k(s_k, \underline{s}_{N\backslash\{k\}}) = (1, p')$.
Now, by using a similar argument as the one in Step 2 by replacing agent 1 by k, we can show that $f_k(\underline{s}) = (1, p')$ which is a contradiction to Step 2.
Step 3.2. Agent $k \in N\backslash\{1\}$ gets the good at s and $s_k = \underline{s}_k$.
We obtain a contradiction using an argument similar to the one in Step 3.1.
Thus, agent 1 gets the good at any s and pays p. ■

Example 7 (continued)
The following Lemma 3 is used in the proof of Proposition 11.

Lemma 3 *For all $s \in \Theta$, $R_k(s'_k, s_{-k})$ is a y-monotonic transform of $R_k(s)$ for all $s'_k > s_k$, $k \in N$ and $y \in A$ such that $y_k = (1, p)$, $p \geq 0$.*

Proof. Take $s \in \Theta$, $k \in N$ and $y \in A$ such that $y_k = (1, p)$, $p \geq 0$ and $s'_k > s_k$. Since g_k is non-decreasing in s_k, $g_k(s'_k, s_{N\backslash\{k\}}) \geq g_k(s)$ which means that agent k values the good in signal profile $(s'_k, s_{N\backslash\{k\}})$ at least as under profile s. Thus, $(1, p)$ weakly improves its position in $R_k(s'_k, s_{N\backslash\{k\}})$ compared to its position in $R_k(s)$. Formally, $R_k(s'_k, s_{N\backslash\{k\}})$ is a y-monotonic transform of $R_k(s)$. ■

Proposition 11 *The environment (Θ, R) in Example 7 is partially knit.*

Proof. Take any two pairs (x, θ), $(z, \widetilde{\theta}) \in A \times \Theta$ such that $\overline{C}(\theta, z, x) \neq \emptyset$, $\#C(\theta, z, x) = 2$. Some agent must get the good either in x or in z, otherwise $\overline{C}(\theta, z, x) = \emptyset$.

First, assume that the same agent i gets the good both in x and in z. Define $\theta' = (\max\{s_i, \widetilde{s}_i\}, \min\{s_j, \widetilde{s}_j\})$, $S = \widetilde{S} = \{\max\{s_i, \widetilde{s}_i\}, \min\{s_j, \widetilde{s}_j\}\}$ where $I(S) = I(\widetilde{S}) = \{i, j\}$. Note that for step $h = 1$, either $s_{i(S,1)} = s_{i(\widetilde{S},1)} = s_i$ if $s_i > \widetilde{s}_i$ or $s_{i(S,1)} = s_{i(\widetilde{S},1)} = \widetilde{s}_i$ if $s_i < \widetilde{s}_i$. Thus, either because there is no signal change or by Lemma 3, we obtain that $R_i(m^1(\theta, S))$ is an x-monotonic transform of $R_i(m^0(\theta, S))$ and $R_i(m^1(\widetilde{\theta}, \widetilde{S}))$ is an z-monotonic transform of $R_i(m^0(\widetilde{\theta}, \widetilde{S}))$. Note that for step 2, either $s_{i(S,h)} = s_{i(\widetilde{S},h)} = s_j$ if $s_j < \widetilde{s}_j$ or $s_{i(S,h)} = s_{i(\widetilde{S},h)} = \widetilde{s}_j$ if $s_j > \widetilde{s}_j$. Thus, either because there is no signal change or by Lemma

2, we obtain in step 2 that $R_j(m^2(\theta,S))$ is an x-monotonic transform of $R_j(m^1(\theta,S))$ and $R_j(m^2(\widetilde{\theta},\widetilde{S}))$ is a z-monotonic transform of $R_j(m^1(\widetilde{\theta},\widetilde{S}))$. Thus, the passage from θ to θ' through S is x-satisfactory, and that from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.

Second, suppose that different agents get the good in x and z. Without loss of generality, say that agent 1 gets the good in x while agent 2 gets it in z. Thus, alternatives x and z are such that $x_1 = (1, p_x)$, $z_1 = (0, 0)$, $x_2 = (0, 0)$, $z_2 = (1, p_z)$.
Now, we consider three cases, and for each one we define θ' and the sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Case 1. $\theta = (0, 1)$.
The conditions $\overline{C}(\theta, z, x) \neq \emptyset$ and $C(\theta, z, x) = N$ are satisfied since $p_x > l$ and $p_z > l$. For any $\widetilde{\theta}$ define $\theta' = \widetilde{\theta}$. If $\widetilde{\theta} = (1, 1)$, let $S = \{\theta_{i(S,1)} = 1\}$, $I(S) = \{1\}$, if $\widetilde{\theta} = (0, 0)$, let $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{2\}$, and if $\widetilde{\theta} = (1, 0)$, let $S = \{\theta_{i(S,1)} = 1, \theta_{i(S,2)} = 0\}$, $I(S) = \{1, 2\}$. By applying Lemma 3, Lemma 2 or both, respectively, we prove that the passage from θ to $\widetilde{\theta} = \theta'$ through S is x-satisfactory.
Case 2. $\theta = (1, 1)$.
For conditions $\overline{C}(\theta, z, x) \neq \emptyset$ and $C(\theta, z, x) = N$ to hold we must have either $p_x > m$ and $p_z \leq m$, or $p_z < m$ and $p_x \geq m$. Suppose that the former holds. Otherwise, a similar proof would follow.
If $\widetilde{\theta} = (0, 1)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$ and $p_z \leq m$.
If $\widetilde{\theta} = (1, 0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{2\}$, and observe that $R_2(m^1(\theta, S))$ is an x-monotonic transform of $R_2(\theta)$ by Lemma 2.
If $\widetilde{\theta} = (0, 0)$, let $\theta' = (0, 1)$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 1\}$, $I(\widetilde{S}) = \{2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$ and $p_z \leq m$. Moreover, $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ since $l < p_z \leq m$.
Case 3. $\theta = (0, 0)$ and $\theta = (1, 0)$.
For both θ, $g_2(\theta) = l$. Since $2 \in C(\theta, z, x)$ then $p_z \leq l$, contradicting our hypothesis.

Third, the last remaining possibility is that in only one of the two alternatives, x or z, some agent gets the good. Without loss of generality, suppose that agent 1 gets the good in x. Note that for conditions $\overline{C}(\theta, z, x) \neq \emptyset$ and $C(\theta, z, x) = N$ to hold, for any $\theta \in \Theta$, $1 \in \overline{C}(\theta, z, x)$ since $2 \in C(\theta, z, x)$.
Now, we consider four cases, and for each one we define θ' and the sequences of types S and $\widetilde{S}$, such that the passage from θ to θ' through S is x-satisfactory and the passage from $\widetilde{\theta}$ to θ' through $\widetilde{S}$ is z-satisfactory.
Case 1. $\theta = (0, 1)$.
Since $1 \in \overline{C}(\theta, z, x)$, $p_x > l$ must be satisfied. For any $\widetilde{\theta}$ define $\theta' = \widetilde{\theta}$. If $\widetilde{\theta} = (1, 1)$, let $S = \{\theta_{i(S,1)} = 1\}$ and $I(S) = \{1\}$, if $\widetilde{\theta} = (0, 0)$, let $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{2\}$, and if $\widetilde{\theta} = (1, 0)$, let $S = \{\theta_{i(S,1)} = 1, \theta_{i(S,2)} = 0\}$ and $I(S) = \{1, 2\}$. By applying either Lemma 3, Lemma 2 or both consecutively in this order, we prove that the passage from θ to $\widetilde{\theta} = \theta'$ through S is x-satisfactory.
Case 2. $\theta = (1, 1)$.

Since $1 \in \overline{C}(\theta, z, x)$, $p_x > m$ must be satisfied.
If $\widetilde{\theta} = (0,1)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$.
If $\widetilde{\theta} = (1,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 1\}$, $I(S) = \{2\}$, and observe that $R_2(m^1(\theta, S))$ is an x-monotonic transform of $R_2(\theta)$ by Lemma 2.
If $\widetilde{\theta} = (0,0)$, let $\theta' = \widetilde{\theta}$ and define $S = \{\theta_{i(S,1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$, $I(S) = \{1,2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$. Moreover, $R_2(m^2(\theta, S))$ is an x-monotonic transform of $R_2(m^1(\theta, S))$ by Lemma 2.
Case 3. $\theta = (0,0)$.
Since $1 \in \overline{C}(\theta, z, x)$, $p_x > l$ must be satisfied.
If $\widetilde{\theta} = (0,1)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$, $I(\widetilde{S}) = \{2\}$, and observe that $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ by Lemma 2.
If $\widetilde{\theta} = (1,0)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$, $I(\widetilde{S}) = \{1\}$, and observe that $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(\widetilde{\theta})$ by Lemma 2.
If $\widetilde{\theta} = (1,1)$, let $\theta' = \theta$ and define $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$, $I(S) = \{2,1\}$, and observe that, by Lemma 2, $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(\widetilde{\theta})$ and $R_1(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$.
Case 4. $\theta = (1,0)$.
Since $1 \in \overline{C}(\theta, z, x)$, $p_x > h$ must be satisfied.
If $\widetilde{\theta} = (0,0)$, let $\theta' = \widetilde{\theta} = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$, $I(S) = \{1\}$, and observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > h$.
If $\widetilde{\theta} = (0,1)$, let $\theta' = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0\}$ and $I(\widetilde{S}) = \{2\}$. Observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > h$. Moreover, $R_2(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ by Lemma 2.
If $\widetilde{\theta} = (1,1)$, let $\theta' = (0,0)$ and define $S = \{\theta_{i(S,1)} = 0\}$ and $I(S) = \{1\}$, $\widetilde{S} = \{\theta_{i(\widetilde{S},1)} = 0, \theta_{i(\widetilde{S},2)} = 0\}$ and $I(\widetilde{S}) = \{1,2\}$. Again, observe that $R_1(m^1(\theta, S))$ is an x-reshuffling of $R_1(\theta)$ since $p_x > m$. Moreover, $R_1(m^1(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_1(\widetilde{\theta})$ and $R_2(m^2(\widetilde{\theta}, \widetilde{S}))$ is a z-monotonic transform of $R_2(m^1(\widetilde{\theta}, \widetilde{S}))$ by Lemma 2. ■

Lemma 4 *$f_{p,p'}$ is non-constant, ex post incentive compatible, and respectful in the environment (Θ, R) in Example 7.*

Proof. By definition $f_{p,p'}$ is not constant. To show that $f_{p,p'}$ is ex post incentive compatible we first observe that agent 1 can never strictly gain by deviating from any $s \in \Theta$. For any $s_2 \in \Theta_2$, since $g_1(0, s_2)) = l$, $g_1(1, s_2) \in \{m, h\}$, and $p \in (l, m)$, then $f_1(0, s_2) P_1(0, s_2) f_1(1, s_2)$ and $f_1(1, s_2) P_1(1, s_2) f_1(0, s_2)$ where $f_1(0, s_2) = (0,0)$ and $f_1(1, s_2) = (1, p)$. Similarly, we can show that agent 2 can never strictly gain by deviating from any $s \in \Theta$. For any $s_1 \in \Theta_1$, since $g_2(s_1, 0) = l$, $g_2(s_1, 1) \in \{m, h\}$, and $p' \in (l, m)$, then $f_2(s_1, 0) R_2(s_1, 0) f_2(s_1, 1)$ and $f_2(s_1, 1) R_2(s_1, 1) f_2(s_1, 0)$ where $f_2(s_1, 0) = (0,0)$ and $f_2(s_1, 1) \in \{(0,0), (1, p')\}$. To check respectfulness, observe that agent 1 is not indifferent between any pair of outcomes obtained when she is the only one changing types. As for agent 2, observe that the same holds if $s_1 = 0$. For $s_1 = 1$, $f(1,0) = f(1,1)$. Thus, respectfulness holds. ■

Remark 2

The following Example and Lemma justify our observation in Remark 2.

Example 8. For simplicity, let $N = \{1,2\}$, $\Theta_i = \{0,1\}$ for all $i \in N$ and $l, m, h \in \mathbb{R}_+$ with $0 = l < m < h$. The agent's preference function is defined as in the general framework but will now be based on a different auxiliary function that takes three possible values, low, medium and high.

More formally,

$$g_i(s) = \begin{cases} l & \text{if} \quad s_i = 0 \\ m & \text{if} \quad s_i = 1 \text{ and } s_j = 0, \\ h & \text{if} \quad s_i = 1 \text{ and } s_j = 1. \end{cases}$$

Observe that for each agent i, g_i satisfies (a) and the following condition:
(e) g_i is non-decreasing in s_j, for all $j \in N \backslash \{i\}$.

Condition (e) establishes that the valuation of the good by agent i depends positively on other agents' signals.

Now, we assert that the environment in Example 8 is neither knit nor partially knit. To do so, we define below a non-constant, ex post incentive compatible and respectful mechanism in such environment that is not ex post group incentive compatible. Therefore, by Theorem 1, the environment is not partially knit.

A mechanism $f_{h,m}$ is such that agent 1 gets the good and pays h if the signal of agent 2 is 0 or both agents' signals are 1 and agent 2 gets the good and pays m otherwise. Formally, for $\theta \in \{0,1\}^2$,

$$f_{h,m}(\theta) = \left\{ \begin{array}{c} ((1,h),(0,0)) \text{ if } s_2 = 0, \\ ((0,0),(1,m)) \text{ if } s_1 = 0,\ s_2 = 1, \text{ and} \\ ((1,h),(0,0)) \text{ if } s_1 = 1 = s_2 \end{array} \right\}.$$

Lemma 5 *$f_{h,m}$ is non-constant, ex post incentive compatible, and respectful but it violates ex post group incentive compatibility in the environment (Θ, R) in Example 8.*

Proof. To check ex post incentive compatibility just observe that no single agent can strictly gain by unilateral deviations. To check respectfulness, we need to consider s and s' such that $f(s) \neq f(s')$, where only one agent changes her type and remains indifferent. Two cases need to be checked. First, let $s = (1,1)$, $s' = (0,1)$. Observe that neither $R_1(0,1)$ is a $f(1,1)$-monotonic transform of $R_1(1,1)$ nor $R_1(1,1)$ is a $f(0,1)$-monotonic transform of $R_1(0,1)$. Second, let $s = (0,1)$ and $s' = (0,0)$. Again, neither $R_2(0,0)$ is a $f(0,1)$-monotonic transform of $R_2(0,1)$ nor $R_2(0,1)$ is a $f(0,0)$-monotonic transform of $R_2(0,0)$. Thus these cases do not need to be considered and respectfulness holds. To check that $f_{h,m}$ violates ex post group incentive compatibility, consider $s = (1,1)$, $C = N$, $s'_C = (0,1)$. Note that $(0,0) = f_1(0,1) I_1(1,1) f_1(1,1) = (1,h)$ and $(1,m) = f_2(0,1) P_2(1,1) f_2(1,1) = (0,0)$ which means that coalition N can ex post profitably deviate under mechanism f at $s \in \Theta$ via s'_C. ■