Equivalence of Piecewise-Linear Approximation and Lagrangian Relaxation for Network Revenue Management

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Equivalence of piecewise-linear approximation and Lagrangian relaxation for network revenue management

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Abstract

The network revenue management (RM) problem arises in airline, hotel, media, and other industries where the sale products use multiple resources. It can be formulated as a stochastic dynamic program, but the dynamic program is computationally intractable because of an exponentially large state space, and a number of heuristics have been proposed to approximate its value function. Notable amongst these—both for their revenue performance, as well as their theoretically sound basis—are approximate dynamic programming methods that approximate the value function by basis functions (both affine functions as well as piecewise-linear functions have been proposed for network RM) and decomposition methods that relax the constraints of the dynamic program to solve simpler dynamic programs (such as the Lagrangian relaxation methods). In this paper we show that these two seemingly distinct approaches coincide for the network RM dynamic program, i.e., the piecewise-linear approximation method and the Lagrangian relaxation method are one and the same.

Key words. network revenue management, linear programming, approximate dynamic programming, Lagrangian relaxation methods.

Revenue management is the control of the sale of a limited quantity of a resource (hotel rooms for a night, airline seats, advertising slots etc.) to a heterogenous population with different valuations for a unit of the resource. The resource is perishable, and for simplicity sake, we assume that it perishes at a fixed point of time in the future. Sale is online, so the firm has to decide what products to offer at a given price for each product, so as not to sell too much at too low a price early and run out of capacity, or, reject too many low-valuation customers and end up with excess unsold inventory.

In industries such as hotels, airlines and media, the products consume bundles of different resources (multi-night stays, multi-leg itineraries) and the decision to accept or reject a particular product at a certain price depends on the future demands and revenues for all the resources used by the product and, indirectly, on all the resources in the network. Network revenue management (network RM) is control based on the demands for the entire network. Chapter 3 of Talluri and van Ryzin [13] contains all the necessary background on network RM.

The network RM problem can be formulated as a stochastic dynamic program, but computing the value function becomes intractable due to the high dimensionality of the state space. As a result,

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researchers have focussed on developing approximation methods. Notable amongst these—both for their revenue performance, as well as their theoretically sound basis—are approximate dynamic programming methods that approximate the value function by basis functions (both affine functions as well as piecewise-linear functions have been proposed for network RM), and decomposition methods that relax the constraints of the dynamic program to solve simpler dynamic programs (such as the Lagrangian methods). In this paper we show that these two seemingly distinct approaches coincide for the network RM dynamic program. Specifically, we show that the piecewise-linear approximation method and the Lagrangian relaxation method are one and the same, using a novel minimal number of binding constraints argument.

As a by-product, we derive some auxiliary results of independent interest: (i) we give a polynomial-time separation procedure for the piecewise-linear approximation linear program, (ii) we show that the optimal solution of the piecewise-linear approximation satisfies monotonicity conditions similar to that of a single-resource dynamic program, and (iii) sketch an extension to a model of network RM with overbooking.

The rest of the paper is organized as follows. In §1 we give a brief survey of the relevant literature. In §2 we first formulate the network RM problem as a dynamic program. We then describe the approximate dynamic programming approach with piecewise-linear basis functions and the Lagrangian relaxation approach. In §3 we give the main body of proofs showing the two approaches are equivalent, along with a simple calculus-based intuition behind our result. In §4 we discuss a choice-model based network RM problem where the equivalence between the two approaches does not hold. This highlights the importance of formulation and customization of the number of Lagrange multipliers to the problem at hand. In §5 we give a small set of numerical results comparing the solution values and running times of the two approaches. In the appendix we give the proofs and also a sketch an extension to an overbooking model of network RM.

1 Relevant literature

Approximate dynamic programming is the name given for methods that replace the value function of a (difficult) dynamic program (DP) with basis functions and solve the simplified problem as an approximation. In this stream of literature, the linear programming approach consists of formulating the dynamic program as a linear program with state-dependent variables representing the value functions and then replacing them by particular functional forms to find the best approximation within that class of functions. In the network RM context, this approach was first investigated by Adelman [1] who uses affine functions; Zhang and Adelman [17] extend this to the choice model of network RM. A natural extension is to use piecewise-linear functions instead of affine as they are very flexible and indeed, for the single-resource dynamic program, optimal. In this vein, Farias and van Roy [4] propose a piecewise-linear approximation with concavity constraints. The resulting linear program has an exponential number of constraints and cannot be solved easily. So they give a constraint sampling heuristic to solve it. Meissner and Strauss [11] extend the piecewise-linear approximation approach to the choice model of network RM using aggregation over states to reduce the number of variables.

Another stream of literature revolves around the Lagrangian relaxation approach to dynamic programming. Here the idea is to relax certain constraints in the dynamic program by associating Lagrange multipliers with them so that the problem decomposes into simpler problems. For network RM, Topaloglu [15] and Kunnumkal and Topaloglu [10] take this approach. Computational results from Topaloglu [15] indicate that Lagrangian relaxation with product-specific Lagrange multipliers gives consistent and clearly superior revenues compared to the other methods, including the affine approximation of Adelman [1]; the piecewise-linear approximation was not included, perhaps because it was not known how to solve it exactly.

How do these seemingly different approaches compare with each other? In a recent paper, Tong and Topaloglu [14] establish the equivalence between the affine approximation of Adelman [1] and the Lagrangian relaxation of Kunnumkal and Topaloglu [10]. Vossen and Zhang [16] give an alternate proof based on Dantzig-Wolfe decomposition. In this paper we show that the piecewise-linear approximation, without the explicit concavity constraints of [4], and the product and time-specific Lagrangian relaxation of Topaloglu [15] coincide; that is, they represent the same linear program. Note that both Adelman [1] and Farias and van Roy [4] formulate the affine and piecewise-linear approximations but leave the question of their tractability open. Indeed the solution methods that they propose (integer-programming, constraint sampling) do not guarantee tractability in theory or scale well in practice. So it is of considerable theoretical and practical interest to show that these strong approximations are also tractable.

Adelman and Mersereau [2] give an equivalence result similar to ours for a class of infinite-horizon discounted restless bandit problems. Infinite-horizon discounted DP problems have a very different flavor than finite-horizon DPs and our techniques are quite different from theirs. We believe the result in this paper is the first such for a non-trivial finite-horizon stochastic DP, and the main implication is that strong tractable approximations for difficult stochastic finite-horizon DPs are indeed possible. The techniques and the intuition developed in this paper can potentially be applied to derive such results; however it appears that even when such a result is true, some amount of customization and problem-specific arguments are required to prove it. For example, for a related problem, namely choice-based network RM, the right formulation and number of Lagrange multipliers to use appear critical. We discuss this further in §4 of this paper; see also [9].

2 Network revenue management

We consider a network RM problem with a set of $\mathcal{I} = \{1, \dots, m\}$ resources (for example flight legs on an airline network), $\mathcal{J} = \{1, \dots, n\}$ products (for example itinerary-fare combinations) that use the resources in $\mathcal{I}$ at the end of τ time periods (booking horizon), with time being indexed from 1 to τ. We assume that each product uses at most one unit of each resource.

2.1 Notation

Throughout, we index resources by i, products by j, and time periods by t. We simplify the notation significantly by making this consistent; for instance if j uses resource i, we represent it as $i \in j$, and all j that use resource i by $\{j \mid j \ni i\}$. We use this notation instead of the somewhat more cumbersome, albeit a bit more precise, option of defining $\mathcal{I}_j \subseteq \mathcal{I}$ as the set of resources used by product j and $\mathcal{J}_i \subseteq \mathcal{J}$ as the set of products that use resource i and then writing $i \in \mathcal{I}_j$ and $j \in \mathcal{J}_i$. We use $\mathbb{1}_{[\cdot]}$ as the indicator function, 1 if true and 0 if false.

Booking requests for products come in over time and we let p_{jt} denote the probability that we get a request for product j at time period t. This is the so-called independent demands model in the revenue management literature. We make the standard assumption that the time periods are small enough so that we get a request for at most one product in each time period. Throughout we assume that $p_{jt} > 0$ for all j. Note that this is without loss of generality because if $p_{jt} = 0$

for some product j, then we can simply discard that product and optimize over a smaller number of products. We also assume that $\sum_j p_{jt} = 1$ for all time periods t. This is also without loss of generality because we can add a dummy product with negligible revenue on each resource. We let f_j denote the revenue associated with product j.

Given a request for product j, the airline has to decide online whether to accept or reject the request. An accepted request generates revenue and consumes capacity on the resources used by the product; a rejected request does not generate any revenue and simply leaves the system.

Throughout, we use boldface for vectors. We represent capacity vectors by $\boldsymbol{r}$. We use superscripts on vectors to index the vectors (for example, the resource capacity vector associated with time period t would be $\boldsymbol{r}^t$) and subscripts to indicate components (for example, the capacity on resource i in time period t would be r_i^t).

We let $\boldsymbol{r}^1 = [r_i^1]$ represent the initial capacity on the resources and $\boldsymbol{r}^t = [r_i^t]$ denote the remaining capacity on resource i at time period t. The remaining capacity r_i^t takes values in the set $\mathcal{R}_i = \{0, \ldots, r_i^1\}$ and $\mathcal{R} = \prod_i \mathcal{R}_i$ represents the state space.

We represent control vectors by $\boldsymbol{u}$. The control vectors could be at the level of the network or at the level of the individual resources. So, $\boldsymbol{u}^t = [u_j^t]$ represents a vector of controls at the network level. We let $u_j^t \in \{0, 1\}$ indicate the acceptance decision—1 if we accept product j at time t and 0 otherwise. On the other hand, $\boldsymbol{u}^{it} = [u_j^{it}]$ represents a control vector associated with resource i at time t. In this case $u_j^{it} \in \{0, 1\}$ represents the acceptance decision on resource i—1 if we accept product j on resource i at time t and 0 otherwise.

2.2 Dynamic Program

The network RM problem can be formulated as a DP. Let

$$\mathcal{U}(\boldsymbol{r}) = \{\boldsymbol{u} \in \{0,1\}^n \,|\, u_j \leq r_i \,\forall j, i \in j\},$$

be the set of *acceptable* products when the state is $\boldsymbol{r}$. The value functions $V_t(\cdot)$ can be obtained through the optimality equations

$$V_t(\boldsymbol{r}) = \max_{\boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})} \sum_j p_{jt} u_j [f_j + V_{t+1}(\boldsymbol{r} - \sum_{i \in j} \boldsymbol{e}^i) - V_{t+1}(\boldsymbol{r})] + V_{t+1}(\boldsymbol{r}), \tag{1}$$

where $\boldsymbol{e}^i$ is a vector with a 1 in the ith position and 0 elsewhere, and the boundary condition is $V_{\tau+1}(\cdot) = 0$. Noting that $\boldsymbol{r}^1$ represents the initial capacity on the resources, $V_1(\boldsymbol{r}^1)$ gives the optimal expected total revenue over the booking horizon.

The value functions can, alternatively, be obtained by solving the linear program with an exponential number of decision variables $\{V_t(\boldsymbol{r}) \,|\, \forall t, \boldsymbol{r} \in \mathcal{R}\}$ and an exponential number of constraints:

$$(DP_{LP}) \qquad \begin{aligned} \min_V \quad & V_1(\boldsymbol{r}^1) \\ \text{s.t} \quad & \\ & V_t(\boldsymbol{r}) \geq \sum_j p_{jt} u_j [f_j + V_{t+1}(\boldsymbol{r} - \sum_{i \in j} \boldsymbol{e}^i) - V_{t+1}(\boldsymbol{r})] + V_{t+1}(\boldsymbol{r}) \\ & \qquad\qquad\qquad\qquad \forall t, \boldsymbol{r} \in \mathcal{R}, \boldsymbol{u} \in \mathcal{U}(\boldsymbol{r}) \\ & V_{\tau+1}(\cdot) = 0. \end{aligned}$$

Both the recursive equations (1) as well as the linear program (DP_{LP}) are intractable. In the following sections, we describe two approximation methods.

2.3 Piecewise-linear approximation for Network RM

We approximate the value functions[1] of (1) by

$$V_t(\boldsymbol{r}) \approx \sum_i v_{it}(r_i), \ \forall \boldsymbol{r} \in \mathcal{R}.$$

Substituting this approximation into (DP_{LP}), we obtain the linear program

$$
(PL) \qquad
\begin{aligned}
V^{PL} = \min_{v} \quad & \textstyle\sum_i v_{i1}(r_i^1) \\
\text{s.t} \quad & \\
& \textstyle\sum_i v_{it}(r_i) \geq \displaystyle\sum_j p_{jt} u_j [f_j + \sum_{i \in j} \{v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)\}] \qquad (2) \\
& \qquad + \sum_i v_{i,t+1}(r_i) \quad \forall t, \boldsymbol{r} \in \mathcal{R}, \boldsymbol{u} \in \mathcal{U}(\boldsymbol{r}) \\
& v_{i,\tau+1}(\cdot) = 0, v_{it}(-1) = \infty, \ \forall t, i,
\end{aligned}
$$

where the decision variables are $\{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$. The number of decision variables in (PL) is $\sum_i r_i^1 \tau$ which is manageable. However, since (PL) has an exponential number of constraints of type (2), we need to use a separation algorithm to generate constraints on the fly to solve (PL) (Grötschel, Lovász, and Schrijver [5]). In §3, we show that the separation can be carried out efficiently for (PL).

A natural question is to understand the structural properties of an optimal solution to (PL). This is useful since imposing structure can often significantly speed up the solution time. Lemma 1 below shows that an optimal solution to (PL) satisfies certain monotonicity properties. In particular, if we interpret $v_{it}(r_i)$ as the value of having r_i units of resource i at time period t, then $v_{it}(r_i) - v_{it}(r_i - 1)$ is the marginal value of capacity on resource i at time period t. Part (i) of Lemma 1 shows that the marginal value of capacity is decreasing in t keeping r_i constant, while part (ii) shows that the marginal value of capacity is decreasing in r_i for a given t.

These properties are quite natural and turn out to be useful for a couple of reasons. First, Lemma 1 implies that the optimal objective function value of (PL) is not affected by adding constraints of the form $v_{it}(r_i) - v_{it}(r_i - 1) \geq v_{it}(r_i + 1) - v_{it}(r_i)$ to the linear program. So the linear program proposed by Farias and van Roy [4], which explicitly imposes the monotonicity constraints, is in fact equivalent to (PL). More importantly, the decreasing marginal value property turns out to be crucial in showing the equivalence between the piecewise-linear approximation and the Lagrangian relaxation approaches.

Lemma 1. *There exists an optimal solution* $\{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ *to* (PL) *such that*
(i) $\hat{v}_{it}(r_i) - \hat{v}_{it}(r_i - 1) \geq \hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1)$ *for all* t, i *and* $r_i \in \mathcal{R}_i$,
(ii) $\hat{v}_{it}(r_i) - \hat{v}_{it}(r_i - 1) \geq \hat{v}_{it}(r_i + 1) - \hat{v}_{it}(r_i)$ *for all* t, i, *and* $r_i \in \mathcal{R}_i$, *where we define* $\hat{v}_{it}(r_i^1 + 1) = \hat{v}_{it}(r_i^1)$ *for all* t *and* i.

Proof. Appendix. □

[1] Adelman [1] uses the *affine* relaxation $V_t(\boldsymbol{r}) \approx \theta_t + \sum_i r_i v_{it}$ but we do not need the offset term θ_t for piecewise-linear approximations as we can use the transformation $\bar{v}_{it}(r_i) = \theta_t/m + v_{it}(r_i)$; see also Adelman and Mersereau [2].

2.4 Lagrangian Relaxation

Topaloglu [15] proposes a Lagrangian relaxation approach that decomposes the network RM problem into a number of single-resource problems by decoupling the acceptance decisions for a product over the resources that it uses via product and time-specific Lagrange multipliers.

Let $\{\lambda_{ijt} \,|\, \forall t, j, i \in j\}$ denote a set of Lagrange multipliers and

$$\mathcal{U}_i(r_i) = \{\boldsymbol{u}^i \in \{0,1\}^n \,|\, u^i_j \leq r_i,\, \forall j \ni i\}$$

denote the set of feasible controls on resource i when its capacity is r_i. We solve the optimality equation

$$\vartheta^\lambda_{it}(r_i) = \max_{\boldsymbol{u}^i \in \mathcal{U}_i(r_i)} \sum_{j \ni i} p_{jt} u^i_j [\lambda_{ijt} + \vartheta^\lambda_{i,t+1}(r_i - 1) - \vartheta^\lambda_{i,t+1}(r_i)] + \vartheta^\lambda_{i,t+1}(r_i)$$

for resource i, with the boundary condition $\vartheta^\lambda_{i,\tau+1}(\cdot) = 0$.

If we define

$$V^\lambda_t(\boldsymbol{r}) = \sum_{t'=t}^{\tau} \sum_j p_{jt'} [f_j - \sum_{i \in j} \lambda_{ijt'}]^+ + \sum_i \vartheta^\lambda_{it}(r_i) \tag{3}$$

it is possible to show that $V^\lambda_t(\boldsymbol{r})$ is an upper bound on $V_t(\boldsymbol{r})$, where we use $[x]^+ = \max\{0, x\}$. The Lagrangian relaxation approach finds the tightest upper bound on the optimal expected revenue by solving

$$V^{LR} = \min_\lambda V^\lambda_1(\boldsymbol{r}^1).$$

Talluri [12] shows that the optimal Lagrange multipliers satisfy $\sum_{i \in j} \lambda_{ijt} = f_j$ for all j and t.

Proposition 1. *There exists* $\{\hat{\lambda}_{ijt} \,|\, \forall t, j, i \in j\} \in \arg\min_\lambda V^\lambda_1(\boldsymbol{r}^1)$ *that satisfy* $\hat{\lambda}_{ijt} \geq 0$ *and* $\sum_{i \in j} \hat{\lambda}_{ijt} = f_j$ *for all* j *and* t.

Proof. Appendix. □

Proposition 1 implies that we can find the optimal Lagrange multipliers by solving

$$V^{LR} = \min_{\{\lambda \,|\, \sum_{i \in j} \lambda_{ijt} = f_j, \lambda_{ijt} \geq 0, \,\forall\, t, j, i \in j\}} \sum_i \vartheta^\lambda_{i1}(r^1_i).$$

Using Proposition 1, we can interpret the Lagrange multiplier λ_{ijt} as the portion of the revenue associated with product j that we allocate to resource i at time period t. With this understanding $\vartheta^\lambda_{i1}(r^1_i)$ is the value function of a single-resource RM problem with revenues $\{\lambda_{ijt} \,|\, \forall j \ni i, t\}$ on resource i. Therefore, we can also obtain the optimal Lagrange multipliers through the linear programming formulation of the single-resource RM dynamic program, with a set of linking constraints

(5) as below:

$$
\begin{aligned}
V^{LR} = \min_{\lambda,\nu} \quad & \sum_i \nu_{i1}(r_i^1) \\
\text{s.t.} \quad & \\
(LR) \qquad & \nu_{it}(r_i) \geq \sum_{j \ni i} p_{jt} u_j^i [\lambda_{ijt} + \nu_{i,t+1}(r_i - 1) - \nu_{i,t+1}(r_i)] \\
& \qquad\qquad + \nu_{i,t+1}(r_i) \quad \forall t, i, r_i \in \mathcal{R}_i, \boldsymbol{u}^i \in \mathcal{U}_i(r_i) \qquad (4) \\
& \sum_{i \in j} \lambda_{ijt} = f_j \quad \forall t, j \qquad (5) \\
& \lambda_{ijt} \geq 0 \; \forall t, j, i \in j; \nu_{i,\tau+1}(\cdot) = 0 \, \forall \, i.
\end{aligned}
$$

The linear programming formulation (LR) turns out to be useful when comparing the Lagrangian relaxation approach with the piecewise-linear approximation.

3 Equivalence of the piecewise-linear approximation and the Lagrangian relaxation approaches

In this section we show that the piecewise-linear approximation and the Lagrangian relaxation approaches are equivalent, in that they yield the same upper bound on the value function. This also shows that the Lagrangian relaxation approach yields the tightest separable, piecewise-linear upper bound to the value function of the network RM dynamic program.

Proposition 2. *$V^{PL} = V^{LR}$.*

It is easy to see that $V^{PL} \leq V^{LR}$ since (LR) gives a separable approximation that is an upper bound, while (PL) gives the tightest separable approximation that is an upper bound; we give a formal proof in §3.2. So the difficult part is the other direction: In the Lagrangian problem, we solve each of the resources independently and a product might be accepted on one resource and rejected on another, and there is no reason to believe that the Lagrange multipliers co-ordinate perfectly—indeed there are few known dynamic programs where they do. For the network RM problem, Proposition 4 below shows that there exists a set of Lagrange multipliers that perfectly coordinate the acceptance decisions across the resources.

In §3.1 we first set up the separation problem for (PL), a simpler alternative to solving (PL) directly. We then show that constraints (2) in (PL) can be separated by solving a linear program. We use this result to prove Proposition 2 in §3.2. We describe a polynomial-time separation algorithm for (PL) in §3.3.

3.1 Separation for (PL)

Since (PL) has an exponential number of constraints of type (2), we use a separation algorithm to solve (PL). Equivalence of efficient separation and solvability of a linear program is due to the well-known work of Grötschel et al. [5].

The idea is to start with a linear program that has a small subset of constraints (2) and solve it to obtain $\bar{\mathcal{V}} = \{\bar{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$. We then check for violated constraints by solving the following separation problem: Prove that $\bar{\mathcal{V}}$ satisfies all the constraints (2), and if not, find a violated constraint. Throughout we assume that $\bar{\mathcal{V}}$ satisfies $\bar{v}_{it}(r_i) - \bar{v}_{it}(r_i - 1) \geq \bar{v}_{it}(r_i + 1) - \bar{v}_{it}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$. This is without loss of generality, since by Lemma 1, we can add these constraints to (PL) without affecting its optimal objective function value.

Let $\Delta_{it}(r_i) = \bar{v}_{i,t+1}(r_i) - \bar{v}_{it}(r_i)$ and $\psi_{it}(r_i) = \bar{v}_{it}(r_i) - \bar{v}_{it}(r_i - 1)$ for $r_i \in \mathcal{R}_i$, so that the separation problem for (PL) for period t can be written as

$$\Phi_t(\bar{\mathcal{V}}) = \max_{\boldsymbol{r} \in \mathcal{R}, \boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})} \sum_j p_{jt} u_j [f_j - \sum_{i \in j} \psi_{i,t+1}(r_i)] + \sum_i \Delta_{it}(r_i). \tag{6}$$

Note that $\psi_{it}(\cdot)$ is just the marginal value of capacity on resource i at time period t. The separation problem for a set of values $\bar{\mathcal{V}}$ is resolved by obtaining the value of $\Phi_t(\bar{\mathcal{V}})$ and checking if for any t, $\Phi_t(\bar{\mathcal{V}}) > 0$.

By Lemma 1, $\psi_{it}(r_i)$ is nonincreasing in r_i. By definition, $\bar{v}_{it}(r_i^1 + 1) = \bar{v}_{it}(r_i^1)$. Since $\psi_{it}(r_i^1) = \bar{v}_{it}(r_i^1) - \bar{v}_{it}(r_i^1 - 1) \geq \bar{v}_{it}(r_i^1 + 1) - \bar{v}_{it}(r_i^1) = 0$, we also have $\psi_{it}(r_i) \geq 0$ for all $r_i \in \mathcal{R}_i$. We show that problem (6) can be solved efficiently as a linear program. This result is useful for two reasons. First, it helps us in establishing the equivalence between the piecewise linear approximation and the Lagrangian relaxation approaches. Second, it shows that separation can be efficiently carried out for (PL).

We begin by describing a relaxation of problem (6) that decomposes it into a number of single resource problems. For time period t, we split the revenue of product j, f_j, among the resources that it consumes using variables λ_{ijt}. We interpret λ_{ijt} as a Lagrange multiplier and it represents the revenue allocated to resource $i \in j$ at time t. As a result, the Lagrange multipliers satisfy $\sum_{i \in j} \lambda_{ijt} = f_j$ and $\lambda_{ijt} \geq 0$ for all $i \in j$, for all j. Given such a set of Lagrange multipliers, we solve the problem

$$\Pi_{it}^{\lambda}(\bar{\mathcal{V}}) = \max_{r_i \in \mathcal{R}_i, \boldsymbol{u}^i \in \mathcal{U}_i(r_i)} \sum_{j \ni i} p_{jt} u_j^i [\lambda_{ijt} - \psi_{i,t+1}(r_i)] + \Delta_{it}(r_i) \tag{7}$$

for each resource i. The following lemma states that $\sum_i \Pi_{it}^{\lambda}(\bar{\mathcal{V}})$ is an upper bound on $\Phi_t(\bar{\mathcal{V}})$.

Lemma 2. *If $\{\lambda_{ijt} \mid \forall t, j, i \in j\}$ satisfy $\sum_{i \in j} \lambda_{ijt} = f_j$ and $\lambda_{ijt} \geq 0$ for all t, j and $i \in j$, then $\Phi_t(\bar{\mathcal{V}}) \leq \sum_i \Pi_{it}^{\lambda}(\bar{\mathcal{V}})$.*

Proof. If $\big(\boldsymbol{r} = [r_i], \boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})\big)$ is optimal for problem (6), then $\boldsymbol{u} \in \mathcal{U}_i(r_i)$ and consequently $(r_i, \boldsymbol{u})$ is feasible for problem (7). □

We next show that the upper bound is tight. That is, letting

$$\Pi_t(\bar{\mathcal{V}}) = \min_{\{\lambda \mid \sum_{i \in j} \lambda_{ijt} = f_j \, \forall j; \lambda_{ijt} \geq 0 \, \forall j, i \in j\}} \sum_i \Pi_{it}^{\lambda}(\bar{\mathcal{V}}) \tag{8}$$

we have the following proposition.

Proposition 3. *$\Phi_t(\bar{\mathcal{V}}) = \Pi_t(\bar{\mathcal{V}})$.*

Before we give a formal proof, we provide some intuition as to why the result holds. The equivalence of $\Phi_t(\bar{\mathcal{V}})$ and $\Pi_t(\bar{\mathcal{V}})$ turns out to be the key result in showing the equivalence between the piecewise-linear and the Lagrangian relaxation approaches. It is quite possible that this type of result applies to other problems (say, to some classes of loosely-coupled dynamic programs, in the framework of Hawkins [6], with a threshold-type or index policy), and we hope the heuristic argument and the intuition we give below will help in such cases.

3.1.1 Intuition behind Proposition 3

Lemma 2 shows that $\Phi_t(\bar{\mathcal{V}}) \leq \Pi_t(\bar{\mathcal{V}})$. So we only give a heuristic argument for why $\Phi_t(\bar{\mathcal{V}}) \geq \Pi_t(\bar{\mathcal{V}})$. Consider problem (6). Noting that $\psi_{i,t+1}(0) = \infty$, an optimal solution will have $u_j = 1$ only if the difference $f_j - \sum_{i\in j} \psi_{i,t+1}(r_i) > 0$. Therefore, we can write $u_j[f_j - \sum_{i\in j} \psi_{i,t+1}(r_i)]$ in the objective function as $[f_j - \sum_{i\in j} \psi_{i,t+1}(r_i)]^+$. Next, recall that $\psi_{i,t+1}(\cdot)$ is a decreasing function of r_i. Assuming it to be invertible (say it is strictly decreasing), we can write problem (6) with $\psi_{i,t+1}$'s as the decision variables instead of the r_i's. Therefore, we can write problem (6) as

$$\Phi_t(\bar{\mathcal{V}}) = \max_{\psi} \sum_j p_{jt}[f_j - \sum_{i\in j} \psi_{i,t+1}]^+ + \sum_i \Delta_{it}(\psi_{i,t+1}). \tag{9}$$

The above problem is not differentiable. However, by smoothing the $[\cdot]^+$ operator and assuming that $\Delta_{it}(\cdot)$ is differentiable, we can solve a differentiable problem which is arbitrarily close to problem (9). So we can assume that an optimizer of the above maximization problem $\{\hat{\psi}_{i,t+1} \,|\, \forall\, i\}$ satisfies the first order condition

$$-\sum_{j\ni i} p_{jt} \mathbb{1}_{[f_j - \sum_{k\in j} \hat{\psi}_{k,t+1} > 0]} + \Delta'_{it}(\hat{\psi}_{i,t+1}) = 0 \tag{10}$$

for all i, where $\Delta'_{it}(\cdot)$ denotes the derivative of $\Delta_{it}(\psi_{i,t+1})$ with respect to $\psi_{i,t+1}$. We emphasize that the above arguments are heuristic; our goal here is to only give intuition.

We use the optimal solution $\{\hat{\psi}_{i,t+1} \,|\, \forall i\}$ described above to construct a set of Lagrange multipliers in the following manner. Let

$$\hat{\lambda}_{ijt} = f_j \frac{\hat{\psi}_{i,t+1}}{\sum_{k\in j} \hat{\psi}_{k,t+1}} \; \forall\, j, i \in j,$$

and note that they are feasible to problem (8). We have

$$\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1} = [f_j - \sum_{k\in j} \hat{\psi}_{k,t+1}] \frac{\hat{\psi}_{i,t+1}}{\sum_{k\in j} \hat{\psi}_{k,t+1}}. \tag{11}$$

Since the ratio on the right hand side is positive, this implies $\mathbb{1}_{[\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1} > 0]} = \mathbb{1}_{[f_j - \sum_{k\in j} \hat{\psi}_{k,t+1} > 0]}$. Since, $\{\hat{\psi}_{i,t+1} \,|\, \forall\, i\}$ satisfies (10), it also satisfies $-\sum_{j\ni i} p_{jt} \mathbb{1}_{[\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1} > 0]} + \Delta'_{it}(\hat{\psi}_{i,t+1}) = 0$, which is the first order condition associated with an optimizer of $\max_\psi \sum_{j\ni i} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}]^+ + \Delta_{it}(\psi_{i,t+1})$. That is, $\{\hat{\psi}_{i,t+1} \,|\, \forall i\}$ is an optimizer of $\max_\psi \sum_{j\ni i} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}]^+ + \Delta_{it}(\psi_{i,t+1})$.

So problem (7) can be written as

$$\Pi^{\hat{\lambda}}_{it}(\bar{\mathcal{V}}) = \max_{\psi} \sum_{j\ni i} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}]^+ + \Delta_{it}(\psi_{i,t+1}) = \sum_{j\ni i} p_{jt}[\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1}]^+ + \Delta_{it}(\hat{\psi}_{i,t+1}), \tag{12}$$

where the last equality follows from above observations.

Putting everything together, we have

$$\begin{aligned}\Pi_t(\bar{\mathcal{V}}) &\leq \sum_i \Pi^{\hat{\lambda}}_{it}(\bar{\mathcal{V}}) = \sum_i \Big\{ \sum_{j\ni i} p_{jt}[\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1}]^+ + \Delta_{it}(\hat{\psi}_{i,t+1}) \Big\} \\ &= \sum_j p_{jt} \sum_{i\in j} [\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1}]^+ + \sum_i \Delta_{it}(\hat{\psi}_{i,t+1}) = \Phi_t(\bar{\mathcal{V}}),\end{aligned}$$

where the first inequality holds since $\{\hat{\lambda}_{ijt} \,|\, \forall i \in j, j\}$ is feasible for problem (8) and the last equality uses (11) and the fact that $\{\hat{\psi}_{i,t+1} \,|\, \forall i\}$ is optimal for (9). Note also that the Lagrange multipliers $\{\hat{\lambda}_{ijt} \,|\, \forall i \in j, j\}$ coordinate the decisions for each product across the different resources: product j is accepted on resource $i \in j$ only if $\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1} > 0$. By (11), either $\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1} > 0$ for all $i \in j$ or $\hat{\lambda}_{ijt} - \hat{\psi}_{i,t+1} \leq 0$ for all $i \in j$. That is, we either accept the product on all the resources it consumes or reject the product on all the resources it consumes.

We once again emphasize that the above arguments are heuristic; we give a formal proof in the following section that is quite distinct from the above reasoning. Nevertheless, it is interesting to note the main conditions for the heuristic argument to work.

1. A threshold type optimal control once we decompose the problem, so the solution can be reconstructed from the Lagrange problem. In this particular case, note that we have $u_j = 1$ only if the revenue f_j exceeds $\sum_{i \in j} \psi_{i,t+1}(r_i)$.
2. A monotone decreasing function (ψ) that allowed us to change the separation problem from optimization over the r's to optimization over the ψ's and use the first-order condition.
3. An adequate number of Lagrange multipliers that allowed us to split the revenue function. In this particular case, we require a Lagrange multiplier λ_{ijt} for each resource i and product j at time t. We discuss this point further in §4.3.

3.1.2 Proof of Proposition 3

We begin with some preliminary results. First, we show that problem (7) can be written as the following linear program

$$
(SepLR_i) \qquad
\begin{aligned}
\Pi^{\lambda}_{it}(\bar{\mathcal{V}}) = \min_{w,z} \quad & w_{it} \\
\text{s.t.} \quad & \\
& w_{it} \geq \sum_{j \ni i} z_{ijtr} + \Delta_{it}(r) \quad \forall r \in \mathcal{R}_i \\
& z_{ijtr} \geq p_{jt}[\lambda_{ijt} - \psi_{i,t+1}(r)] \quad \forall j \ni i, r \in \mathcal{R}_i \\
& z_{ijtr} \geq 0 \, \forall j \ni i, r \in \mathcal{R}_i.
\end{aligned}
$$

Lemma 3. *The linear program ($SepLR_i$) is equivalent to (7).*

Proof. Appendix. □

We can, therefore, formulate problem (8) as the linear program

$$
(SepLR) \quad \begin{aligned}
\Pi_t(\bar{\mathcal{V}}) = \min_{\lambda,w,z} \quad & \sum_i w_{it} \\
\text{s.t.} \quad & \\
& w_{it} \geq \sum_{j \ni i} z_{ijtr} + \Delta_{it}(r) \quad \forall i, r \in \mathcal{R}_i && (13)\\
& z_{ijtr} \geq p_{jt}[\lambda_{ijt} - \psi_{i,t+1}(r)] \quad \forall i, j \ni i, r \in \mathcal{R}_i && (14)\\
& \sum_{i \in j} \lambda_{ijt} = f_j \quad \forall j && (15)\\
& \lambda_{ijt} \geq 0 \quad \forall i, j \ni i && (16)\\
& z_{ijtr} \geq 0 \quad \forall i, j \ni i, r \in \mathcal{R}_i. && (17)
\end{aligned}
$$

With a slight abuse of notation, we let $(\lambda, w, z) = \big(\{\lambda_{ijt}v | \forall j, i \in j\}, \{w_{it} \,|\, \forall i\}, \{z_{ijtr} \,|\, \forall i, j \ni i, r \in \mathcal{R}_i\}\big)$ denote a feasible solution to $(SepLR)$. Let $\xi_{it}(r) = w_{it} - [\sum_{j \ni i} z_{ijtr} + \Delta_{it}(r)]$ denote the slack in constraint (13), and $B_i(\lambda, w, z) = \{r \in \mathcal{R}_i \,|\, \xi_{it}(r) = 0\}$ denote the set of binding constraints of type (13) and $B_i^c(\lambda, w, z)$ denote its complement.

Note that if $(\hat{\lambda}, \hat{w}, \hat{z})$ is an optimal solution, then $B_i(\hat{\lambda}, \hat{w}, \hat{z})$ is nonempty, since for each resource i, there exists some $r_i \in \mathcal{R}_i$ such that constraint (13) is satisfied as an equality.

The following proposition is a key result. It says that there exists a set of optimal Lagrange multipliers that perfectly coordinate the acceptance decisions for each product across all the resources. That is, even though we solve the single resource problems in a decentralized fashion, the Lagrange multipliers are such that we either accept the product on all the resources or reject it on all the resources.

We say that $(\hat{\lambda}, \hat{w}, \hat{z})$ is an optimal solution to $(SepLR)$ with a *minimal set of binding constraints* $\bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z})$, if there is no other optimal solution $(\hat{\lambda}', \hat{w}', \hat{z}')$ which has a set of binding constraints that is a strict subset of the binding constraints of $(\hat{\lambda}, \hat{w}, \hat{z})$; that is,

$$
\bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}', \hat{w}', \hat{z}') \subsetneq \bigcup_{i \in \mathcal{I}} B_i(\hat{\lambda}, \hat{w}, \hat{z}).
$$

Proposition 4. *There exists an optimal solution $(\hat{\lambda}, \hat{w}, \hat{z})$ to (SepLR) with a minimal set of binding constraints and $\{\hat{r}_i \,|\, \hat{r}_i \in B_i(\hat{\lambda}, \hat{w}, \hat{z}), \forall i\}$ such that for each j, we either have $\hat{\lambda}_{ijt} \leq \psi_{i,t+1}(\hat{r}_i)$ for all $i \in j$ or $\hat{\lambda}_{ijt} \geq \psi_{i,t+1}(\hat{r}_i)$ for all $i \in j$.*

Proof. Appendix. □

By Lemma 2, $\Phi_t(\bar{\mathcal{V}}) \leq \Pi_t(\bar{\mathcal{V}})$. We show below that $\Phi_t(\bar{\mathcal{V}}) \geq \Pi_t(\bar{\mathcal{V}})$, which completes the proof. Let $(\hat{\lambda}, \hat{w}, \hat{z})$ and $\{\hat{r}_i \,|\, \forall i\}$ be as in Proposition 4; so we have

$$
\Pi_t(\bar{\mathcal{V}}) = \sum_i \hat{w}_{it} = \sum_i \sum_{j \ni i} \hat{z}_{i,j,t,\hat{r}_i} + \Delta_{it}(\hat{r}_i) = \sum_j \sum_{i \in j} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)]^+ + \sum_i \Delta_{it}(\hat{r}_i),
$$

where the second equality holds since $\hat{r}_i \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$ for all i. The last equality holds since if $\hat{z}_{i,j,t,\hat{r}_i} > p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)]^+$, then we can decrease $\hat{z}_{i,j,t,\hat{r}_i}$ by a small positive number contradicting either the optimality of $(\hat{\lambda}, \hat{w}, \hat{z})$ or the fact that $(\hat{\lambda}, \hat{w}, \hat{z})$ is an optimal solution with a minimal set of binding constraints amongst all optimal solutions. Let $\mathcal{J}_1 = \{j \,|\, \hat{\lambda}_{ijt} \geq \psi_{i,t+1}(\hat{r}_i) \forall i \in j\}$ and

$\mathcal{J}_2 = \mathcal{J} \backslash \mathcal{J}_1$ where $\mathcal{J} = \{1, \ldots, n\}$. By Proposition 4, every product $j \in \mathcal{J}_2$ satisfies $\hat{\lambda}_{ijt} \leq \psi_{i,t+1}(\hat{r}_i)$ for all $i \in j$. Therefore,

$$\begin{aligned}
\Pi_t(\bar{\mathcal{V}}) &= \sum_{j \in \mathcal{J}_1} \sum_{i \in j} p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)] + \sum_i \Delta_{it}(\hat{r}_i) \\
&= \sum_j \sum_{i \in j} p_{jt} \hat{u}_j [\hat{\lambda}_{ijt} - \psi_{i,t+1}(\hat{r}_i)] + \sum_i \Delta_{it}(\hat{r}_i) \\
&= \sum_j p_{jt} \hat{u}_j [f_j - \sum_{i \in j} \psi_{i,t+1}(\hat{r}_i)] + \sum_i \Delta_{it}(\hat{r}_i) \\
&\leq \Phi_t(\bar{\mathcal{V}})
\end{aligned}$$

where we define $\hat{u}_j = 1$ for $j \in \mathcal{J}_1$ and $\hat{u}_j = 0$ for $j \in \mathcal{J}_2$. Note that the last equality follows from constraint (15). The last inequality holds since $\boldsymbol{r} = [\hat{r}_i], \boldsymbol{u} = [\hat{u}_j]$ is feasible to problem (6) by the following argument: we trivially have $\hat{u}_j \leq \hat{r}_i$ for all $j \in \mathcal{J}_2$ and $i \in j$. On the other hand, since $\hat{\lambda}_{ijt}$ is finite and $\psi_{i,t+1}(0) = \infty$, we have $\hat{r}_i \geq 1$ for all $j \in \mathcal{J}_1$ and $i \in j$. It follows that $\hat{u}_j \leq \hat{r}_i$ for all $j \in \mathcal{J}_1$ and $i \in j$; so the 0-1 controls $\hat{\boldsymbol{u}} = \{\hat{u}_j\}$ satisfy the constraint that $\hat{u}_j = 0$ if $\hat{r}_i = 0$ for any $i \in j$. □

3.2 Proof of Proposition 2

We first show that $V^{PL} \leq V^{LR}$. Consider a feasible solution $\big(\{\hat{\lambda}_{ijt} \,|\, \forall t, j, i \in j\}, \{\hat{\nu}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}\big)$ to (LR). For a given $t, \boldsymbol{r} = [r_i]$ and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})$, note that $\boldsymbol{u} \in \mathcal{U}_i(r_i)$ for all i. Summing up constraints (4) for r_i and $\boldsymbol{u} \in \mathcal{U}_i(r_i)$ for all i,

$$\begin{aligned}
\sum_i \hat{\nu}_{it}(r_i) &\geq \sum_i \sum_{j \ni i} p_{jt} u_j [\hat{\lambda}_{ijt} + \hat{\nu}_{i,t+1}(r_i - 1) - \hat{\nu}_{i,t+1}(r_i)] + \sum_i \hat{\nu}_{i,t+1}(r_i) \\
&= \sum_j p_{jt} u_j [f_j + \sum_{i \in j} \hat{\nu}_{i,t+1}(r_i - 1) - \hat{\nu}_{i,t+1}(r_i)] + \sum_i \hat{\nu}_{i,t+1}(r_i)
\end{aligned}$$

where the equality holds since $\sum_{i \in j} \hat{\lambda}_{ijt} = f_j$. So $\{\hat{\nu}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ is a feasible solution to (PL) with the same objective function value and we have $V^{PL} \leq V^{LR}$.

To show the reverse inequality, let $\underline{V}^{PL} = \min_v \sum_i v_{i1}(r_i^1) + \sum_t \Pi_t(\mathcal{V})$, where $\mathcal{V} = \{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$. We have

$$V^{PL} = \min_{\{v \,|\, \Phi_t(\mathcal{V}) \leq 0\}} \sum_i v_{i1}(r_i^1) = \min_{\{v \,|\, \Pi_t(\mathcal{V}) \leq 0\}} \sum_i v_{i1}(r_i^1) \geq \min_{\{v | \Pi_t(\mathcal{V}) \leq 0\}} \sum_i v_{i1}(r_i^1) + \sum_t \Pi_t(\mathcal{V}) \geq \underline{V}^{PL},$$

where the first equality follows from (6) while the second one follows from Proposition 3. The first inequality follows since $\Pi_t(\mathcal{V})$ is constrained to be nonpositive, while the last equality uses the fact that $\underline{V}^{PL}$ does not have the constraints $\Pi_t(\mathcal{V}) \leq 0$.

Using (7) and the fact that $\Pi_{it}^{\lambda}(\mathcal{V})$ appears in the objective function of a minimization problem, we have

$$
\begin{aligned}
\underline{V}^{PL} &= \min_{\lambda,\pi,v} \sum_t \sum_i \pi_{it} + \sum_i v_{i1}(r_i^1) \\
\text{s.t.} & \\
& \pi_{it} \geq \sum_{j \ni i} p_{jt} u_j^i [\lambda_{ijt} + v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] + v_{i,t+1}(r_i) - v_{it}(r_i) \\
& \qquad\qquad \forall t, i, r_i \in \mathcal{R}_i, \boldsymbol{u}^i \in \mathcal{U}_i(r_i) \\
& \sum_{i \in j} \lambda_{ijt} = f_j \quad \forall t, j \\
& \lambda_{ijt} \geq 0 \, \forall t, j, i \in j; v_{i,\tau+1}(\cdot) = 0 \, \forall i.
\end{aligned}
$$

Letting $\pi_{it} = \theta_{it} - \theta_{i,t+1}$ with $\theta_{i,\tau+1} = 0$, the above objective function becomes $\sum_i \theta_{i1} + v_{i1}(r_i^1)$ while the first set of constraints become $\theta_{it} + v_{it}(r_i) \geq \sum_{j \ni i} p_{jt} u_{ij} [\lambda_{ijt} + v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] + \theta_{i,t+1} + v_{i,t+1}(r_i)$. Finally, letting $\nu_{it}(r_i) = \theta_{it} + v_{it}(r_i)$, we have

$$
\begin{aligned}
\underline{V}^{PL} &= \min_{\lambda,\nu} \sum_i \nu_{i1}(r_i^1) \\
\text{s.t.} & \\
& \nu_{it}(r_i) \geq \sum_{j \ni i} p_{jt} u_j^i [\lambda_{ijt} + \nu_{i,t+1}(r_i - 1) - \nu_{i,t+1}(r_i)] + \nu_{i,t+1}(r_i) \\
& \qquad\qquad \forall t, i, r_i \in \mathcal{R}_i, \boldsymbol{u}^i \in \mathcal{U}_i(r_i) \\
& \sum_{i \in j} \lambda_{ijt} = f_j \quad \forall t, j \\
& \lambda_{ijt} \geq 0 \, \forall t, j, i \in j; \nu_{i,\tau+1}(\cdot) = 0 \, \forall i.
\end{aligned}
$$

The above linear program is exactly (LR), the linear programming formulation of the Lagrangian relaxation. So $V^{LR} = \underline{V}^{PL} \leq V^{PL}$.

Therefore $V^{LR} = V^{PL}$ as we argued the other, easier, direction earlier. □

3.3 Polynomial-time separation for (PL)

The separation for (PL) can be done by solving the compact linear program $(SepLR)$ for a given set of $\bar{\mathcal{V}}$ variables. If its optimal objective function value $\Pi_t(\bar{\mathcal{V}}) \leq 0$ for all t then $\bar{\mathcal{V}}$ is feasible in (PL). If $\Pi_t(\bar{\mathcal{V}}) > 0$ for some t, then we find a state-action pair $(\boldsymbol{r} \in \mathcal{R}, \boldsymbol{u} \in \mathcal{U}(\boldsymbol{r}))$ that violates constraint (2) in the following manner.

Separation Algorithm

Step 1: Let $(\hat{\lambda}^{(0)}, \hat{w}^{(0)}, \hat{z}^{(0)})$ be an optimal solution to $(SepLR)$. Set $k = 0$.

Step 2: Let $\{\hat{r}_i^{(k)} \,|\, \forall i\}$ be as defined in Proposition 4.
If, for all j, $\hat{\lambda}_{ijt}^{(k)} \leq \psi_{i,t+1}(\hat{r}_i^{(k)})$ for all $i \in j$ or $\hat{\lambda}_{ijt}^{(k)} \geq \psi_{i,t+1}(\hat{r}_i^{(k)})$ for all $i \in j$, set $u_j = 1$ for all $j \in \mathcal{J}_1$ and $u_j = 0$ for all $j \in \mathcal{J}_2$, where $\mathcal{J}_1$ and $\mathcal{J}_2$ are as defined in Proposition 3. Set $\boldsymbol{r} = \{\hat{r}_i^{(k)} \,|\, \forall i\}$ and $\boldsymbol{u} = \{u_j \,|\, \forall j\}$ and stop.
Else, pick a product j such that for $i \in j$, we have $\hat{\lambda}_{ijt}^{(k)} < \psi_{i,t+1}(\hat{r}_i^{(k)})$, while for $l \in j$, we have $\hat{\lambda}_{ljt}^{(k)} > \psi_{l,t+1}(\hat{r}_l^{(k)})$. Let $(\bar{\lambda}^{(k)}, \bar{w}^{(k)}, \bar{z}^{(k)})$ be as described in Proposition 4.

Step 3: Set $\hat{\lambda}^{(k+1)} = \bar{\lambda}^{(k)}, \hat{w}^{(k+1)} = \bar{w}^{(k)}$ and $\hat{z}^{(k+1)} = \bar{z}^{(k)}$. Set $k = k + 1$ and go to Step 2.

By Proposition 4, $(\hat{\lambda}^{(k)}, \hat{w}^{(k)}, \hat{z}^{(k)})$ is an optimal solution to $(SepLR)$ for all k. Proposition 4 also implies that $(\hat{\lambda}^{(k+1)}, \hat{w}^{(k+1)}, \hat{z}^{(k+1)})$ has strictly fewer number of binding constraints of type (13) than $(\hat{\lambda}^{(k)}, \hat{w}^{(k)}, \hat{z}^{(k)})$. Since the number of binding constraints of type (13) in any optimal solution is at least m and at most $\sum_i r_i^1$, Separation Algorithm terminates in polynomial time. Finally, by Proposition 3, $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})$.

4 Network RM with customer choice—failure of the equivalence

We consider the network RM problem with customer choice behavior. In contrast to the independent demands setting described earlier, customers do not come in with the intention of purchasing a fixed product; rather their purchasing decision is influenced by the set of products that are made available for sale. This problem appears to be an order of magnitude more difficult to approximate than the independent-class network RM problem. Indeed, in Kunnumkal and Talluri [8] we show that even the affine approximation of the dynamic program under the simplest possible choice model, a single-segment multinomial-logit model, is NP-complete.

In choice-based RM, a customer chooses product j with probability $p_j(S)$, when S is the set of products offered. Note that $p_j(S) = 0$ if $j \notin S$ and $1 - \sum_j p_j(S)$ is the probability that the customer does not choose any of the offered products. Letting $\mathcal{Q}(\boldsymbol{r}) = \{j \,|\, \mathbb{1}_{[j \ni i]} \leq r_i \,\forall i\}$ denote the set of products that can be offered given the resource capacities $\boldsymbol{r}$, the value functions $V_t(\cdot)$ can be obtained through the optimality equations

$$V_t(\boldsymbol{r}) \quad = \quad \max_{S \subset \mathcal{Q}(\boldsymbol{r})} \sum_j p_j(S)[f_j + V_{t+1}(\boldsymbol{r} - \sum_{i \in j} \boldsymbol{e}^i) - V_{t+1}(\boldsymbol{r})] + V_{t+1}(\boldsymbol{r}),$$

and the boundary condition is $V_{\tau+1}(\cdot) = 0$.

The value functions can alternatively be obtained by solving the linear program

$$(CDP_{LP}) \qquad \begin{aligned} \min_{V_t(\cdot)} \quad & V_1(\boldsymbol{r}^1) \\ \text{s.t} \quad & \\ & V_t(\boldsymbol{r}) \geq \sum_j p_j(S)[f_j + V_{t+1}(\boldsymbol{r} - \sum_{i \in j} \boldsymbol{e}^i) - V_{t+1}(\boldsymbol{r})] + V_{t+1}(\boldsymbol{r}) \\ & \qquad\qquad\qquad\qquad \forall t, \boldsymbol{r} \in \mathcal{R}, S \subset \mathcal{Q}(\boldsymbol{r}) \\ & V_{\tau+1}(\cdot) = 0. \end{aligned}$$

Computing the value functions either through the optimality equations or the linear program is intractable. In the following sections, we describe extensions of the piecewise-linear and Lagrangian relaxation approaches to choice-based RM.

4.1 Piecewise-linear approximation

The linear program from using a separable piecewise-linear approximation to the value function $V_t(\boldsymbol{r}) \approx \sum_i v_{it}(r_i)$ for all $\boldsymbol{r} \in \mathcal{R}$ is

$$(CPL) \qquad \begin{aligned} V^{CPL} = \min_{v} \quad & \textstyle\sum_i v_{i1}(r_i^1) \\ \text{s.t} \quad & \\ & \textstyle\sum_i v_{it}(r_i) \geq \sum_j p_j(S)\big[f_j + \sum_{i \in j}\{v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)\}\big] \\ & \qquad + \sum_i v_{i,t+1}(r_i) \quad \forall t, \boldsymbol{r} \in \mathcal{R}, S \subset \mathcal{Q}(\boldsymbol{r}) \\ & v_{i,\tau+1}(\cdot) = 0 \quad \forall i. \end{aligned} \qquad (18)$$

Meissner and Strauss [11] propose the piecewise-linear approximation for choice-based network RM. In order to make the formulation more tractable, they use an aggregation over the state variables to reduce to number of variables.

4.2 Lagrangian Relaxation

A natural extension of the Lagrangian relaxation approach to choice-based RM is to use product and time-specific Lagrange multipliers to decompose the network problem into a number of single resource problems. We show that the Lagrangian relaxation approach using product and time-specific multipliers turns out to be weaker than the piecewise-linear approximation. The number of Lagrange multipliers that we use appears to be crucial to get the equivalence between the Lagrangian relaxation and the piecewise-linear approximation.

4.2.1 Lagrangian relaxation using product-specific multipliers

A natural extension of the Lagrangian relaxation approach to the choice-based network RM problem is to use product and time-specific Lagrange multipliers $\{\lambda_{ijt} \,|\, \forall t, j, i \in j\}$ to decompose the network problem into a number of single resource problems. Letting $\mathcal{Q}_i(r_i) = \{j \,|\, \mathbb{1}_{[j \ni i]} \leq r_i\}$, we solve the optimality equation

$$\vartheta_{it}^{\lambda}(r_i) = \max_{S \subset \mathcal{Q}_i(r_i)} \sum_{j \ni i} p_j(S)[\lambda_{ijt} + \vartheta_{i,t+1}^{\lambda}(r_i - 1) - \vartheta_{i,t+1}^{\lambda}(r_i)] + \vartheta_{i,t+1}^{\lambda}(r_i)$$

for resource i. It is possible to show that $V_t^{\lambda}(\boldsymbol{r}) = \sum_i \vartheta_{it}^{\lambda}(r_i)$ is an upper bound on $V_t(\boldsymbol{r})$. We find the tightest upper bound on the optimal expected revenue by solving

$$V^{CLRp} = \min_{\{\lambda \mid \lambda_{ijt} \geq 0, \sum_{i \in j} \lambda_{ijt} = f_j \ \forall j,t\}} V_1^{\lambda}(\boldsymbol{r}^1).$$

In contrast to the independent demands setting, the example below illustrates that we can have $V^{CPL} < V^{CLRp}$.

4.2.2 Example where $V^{CPL} < V^{CLRp}$

Consider a network RM problem with two products, two resources and a single time period in the booking horizon. The first product uses only the first resource, while the second product uses only the second resource, and we have a single unit of capacity on each resource. Note that in the airline context, this example corresponds to a parallel flights network. The revenues associated with the products are $f_1 = 10$ and $f_2 = 1$. The choice probabilities are given in Table 1. In the Lagrangian relaxation, since we have Lagrange multipliers only for $j \ni i$, we have only two multipliers $\lambda_{1,1,1}$ and $\lambda_{2,2,1}$. Moreover, the constraint $\sum_{i \in j} \lambda_{ijt} = f_j$ implies that all feasible Lagrange multipliers satisfy $\lambda_{1,1,1} = f_1$ and $\lambda_{2,2,1} = f_2$. Noting that there is only a single time period in the booking horizon and that $\mathcal{Q}_i(1) = \mathcal{J}$ for $i = 1, 2$,

$$\vartheta_{11}^{\lambda}(1) = \max_{S \subset \mathcal{J}} p_1(S) f_1 = 5$$

and

$$\vartheta_{21}^{\lambda}(1) = \max_{S \subset \mathcal{J}} p_2(S) f_2 = 10/11$$

so that $V^{CLRp} = 65/11$.

Letting $S_1 = \{1\}, S_2 = \{2\}$ and $S_3 = \{1, 2\}$, the linear program associated with the piecewise-linear approximation is

$$(CPL) \qquad \begin{aligned} V^{CPL} = \min_{v} \quad & v_{11}(1) + v_{21}(1) \\ \text{s.t} \quad & \\ & v_{11}(1) + v_{21}(1) \geq \max\{p_1(S_1)f_1, p_2(S_2)f_2, p_1(S_3)f_1 + p_2(S_3)f_2, 0\} \\ & v_{11}(1) + v_{21}(0) \geq \max\{p_1(S_1)f_1, 0\} \\ & v_{11}(0) + v_{21}(1) \geq \max\{p_2(S_2)f_2, 0\} \\ & v_{11}(0) + v_{21}(0) \geq 0. \end{aligned}$$

Note that the first, second, third and fourth constraints correspond to the states vectors $[1, 1]$, $[1, 0]$, $[0, 1]$ and $[0, 0]$, respectively. It is easy to verify that an optimal solution to (CPL) is $v_{11}(1) = 5, v_{11}(0) = 10/11, v_{21}(1) = 0, v_{21}(0) = 0$ and we have $V^{CPL} = 5 < V^{CLRp}$. Therefore, the Lagrangian relaxation approach with product and time-specific Lagrange multipliers is weaker than the piecewise-linear approximation for choice-based network RM.

S	$p_1(S)$	$p_2(S)$
$\{1\}$	1/2	0
$\{2\}$	0	10/11
$\{1,2\}$	1/12	10/12

Table 1: Choice probabilities for the example where $V^{CPL} < V^{CLRp}$.

4.3 Lagrangian relaxation using offer set-specific multipliers

Next we consider a Lagrangian relaxation approach with an expanded set of multipliers. We introduce some notation first. For a resource i and offer set S, we write $i \in S$ if there exists a product $j \in S$ with $j \ni i$. We interpret $S \ni i$ in a similar fashion. Letting

$$R(S) = \sum_j p_j(S) f_j \tag{19}$$

be the revenue associated with offer set S, we let λ_{iSt} be the portion allocated to resource $i \in S$ and $\lambda_{\phi St} = R(S) - \sum_{i \in S} \lambda_{iSt}$ denote the difference between the revenue associated with the offer set S and the allocations across the resources. Given a set of Lagrange multipliers $\{\lambda_{\phi St}, \lambda_{iSt} \,|\, \forall t, S, i \in S\}$, we solve the optimality equation

$$\vartheta^\lambda_{it}(r_i) = \max_{S \subset \mathcal{Q}_i(r_i)} \mathbb{1}_{[S \ni i]} \lambda_{iSt} + \sum_{j \ni i} p_j(S)[\vartheta^\lambda_{i,t+1}(r_i - 1) - \vartheta^\lambda_{i,t+1}(r_i)] + \vartheta^\lambda_{i,t+1}(r_i)$$

for resource i, with the boundary condition that $\vartheta^\lambda_{i,\tau+1}(\cdot) = 0$. Letting

$$\vartheta^\lambda_{\phi t} = \max_{S \subset 2^{\mathcal{J}}} \lambda_{\phi St} + \vartheta^\lambda_{\phi,t+1},$$

where $2^{\mathcal{J}}$ denotes the collection of all subsets of $\mathcal{J}$ and $\vartheta^\lambda_{\phi,\tau+1} = 0$, it is possible to show that $V^\lambda_t(\boldsymbol{r}) = \sum_i \vartheta^\lambda_{it}(r_i) + \vartheta^\lambda_{\phi t}$ is an upper bound on $V_t(\boldsymbol{r})$. We find the tightest upper bound on the optimal expected revenue by solving (say as a linear program)

$$V^{CLRo} = \min_{\{\lambda | \lambda_{\phi St} + \sum_{i \in S} \lambda_{iSt} = R(S) \,\forall\, S, t\}} V^\lambda_1(\boldsymbol{r}^1).$$

In Kunnumkal and Talluri [9] we show the following using a more elaborate version of the minimal-binding-constraints argument employed here:

Proposition 5. *([9])* $V^{CPL} = V^{CLRo}$.

4.4 Number of Lagrange multipliers

Compared to the independent demands case considered previously, the difference in the choice setting is that the revenue associated with an offer set S, $R(S)$, is much more complicated than before and is not naturally given as a sum of expected revenues from each product. This is what precludes equivalence with product-specific multipliers.

In fact, requiring $\sum_{i \in S} \lambda_{iSt} = R(S)$ can be overly restrictive also and we do not necessarily have $V^{CPL} = V^{CLRo}$, if we impose this constraint on the Lagrange multipliers. We illustrate with an example. Consider a network revenue management problem with two products, two resources and a single time period in the booking horizon. The first product uses only the first resource, while the second product uses only the second resource, and we have a single unit of capacity on each resource. Note that in the airline context, this example corresponds to a parallel flights network.

S	$p_1(S)$	$p_2(S)$
$\{1\}$	$50/99$	0
$\{2\}$	0	$51/101$
$\{1,2\}$	$1/2$	$1/2$

Table 2: Choice probabilities for the example $V^{CPL} < \min_{\lambda | \sum_{i\in S} \lambda_{iSt} = R(S), \forall S,t} V_1^{\lambda}(\boldsymbol{r}^1)$.

The revenues associated with the products are $f_1 = 99$ and $f_2 = 101$. The choice probabilities are given in Table 2. Letting $S_1 = \{1\}, S_2 = \{2\}$ and $S_3 = \{1,2\}$, we have $R(S_1) = 50$, $R(S_2) = 51$ and $R(S_3) = 100$. If we impose the constraint $\sum_{i\in S} \lambda_{iSt} = R(S)$ on the Lagrange multipliers, then we have $\lambda_{1,S_1,1} = 50, \lambda_{2,S_2,1} = 51$ and $\lambda_{1,S_3,1} + \lambda_{2,S_3,1} = 100$ for all feasible Lagrange multipliers. We have $\vartheta_{11}^{\lambda}(1) = \max\{50, \lambda_{1,S_3,1}\}$ and $\vartheta_{21}^{\lambda}(1) = \max\{51, \lambda_{2,S_3,1}\}$. It can be verified that

$$\min_{\{\lambda | \lambda_{1,S_1,1}=50, \lambda_{2,S_2,1}=51, \lambda_{1,S_3,1}+\lambda_{2,S_3,1}=100\}} \vartheta_{11}^{\lambda}(1) + \vartheta_{21}^{\lambda}(1) = 101 > V^{CPL} = 100.$$

The optimal solution can be reached by the Lagrangian if we introduce a multiplier $\lambda_{\phi St}$ for each S: Set $\lambda_{\phi,S_1,t} = \lambda_{\phi,S_2,t} = \lambda_{\phi,S_3,t} = 1$ in the above problem, and the Lagrangian relaxation V^{CLRo} achieves the value 100. Therefore, the choice as well as the number of Lagrangian multipliers seems critical and specific to each problem.

5 Numerical results

As the separation problem for (PL) is solvable in polynomial time, a plausible solution procedure is by linear programming, generating the constraints on the fly. In this section, we investigate how (PL) compares (LR). By our theoretical result both should give the same objective function value (as the numerical results indeed show), so our main interest is in comparing solution times.

We consider a hub and spoke network with a single hub that serves N spokes. There is one flight from the hub to each spoke and one flight from each spoke to the hub. The total number of flights is $2N$. Note that the flight legs correspond to the resources in our network RM formulation. We have high fare-product and low fare-product connecting each origin-destination pair. Consequently, there are $2N(N+1)$ fare-products in total. In all of our test problems, the high fare-product connecting an origin-destination pair is twice as expensive as the corresponding low fare-product. We measure the tightness of the flight leg capacities by

$$\alpha = \frac{\sum_i \sum_t \sum_{j \ni i} p_{jt}}{\sum_i r_i^1},$$

where the numerator measures the total expected demand over the flight legs and the denominator gives the sum of the capacities of the flight legs. We index the test problems using (τ, N, α), where τ is the number of periods in the booking horizon and N and α are as defined above. We use $\tau \in \{25, 50, 100\}$, $N \in \{2,3,4\}$ and $\alpha \in \{1.0, 1.2, 1.6\}$ so that we get a total of 27 test problems. We note that our test problems are adapted from those in Topaloglu [15].

We use constraint generation to solve (PL). By Lemma 1 we can add constraints of the form $v_{it}(r_i) - v_{it}(r_i - 1) \geq v_{it}(r_i + 1) - v_{it}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$ to (PL) without affecting its optimal objective function value. So, while solving (PL), we start with a linear program that only has the above mentioned constraints and the nonnegativity constraints. We add constraints of type (2) on the fly by solving the separation problem described in §3.1. By Proposition 3 and Lemma 3, we can solve the separation problem as a linear program. We add violated constraints to (PL) and stop when we are within 1% of optimality.

We use constraint generation to solve (LR) as well. It can be verified that there exists an optimal solution $\{\hat{\nu}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ to (LR) that satisfies $\hat{\nu}_{it}(r_i) - \hat{\nu}_{it}(r_i - 1) \geq \hat{\nu}_{it}(r_i + 1) - \hat{\nu}_{it}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$. While solving (LR), we start with a linear program that only has the above mentioned constraints in addition to constraints (5) and the nonnegativity constraints. We add constraints of type (4) on the fly by solving the following separation problem. Given a solution $\big(\{\hat{\lambda}_{ijt} \,|\, \forall t, j, i \in j\}, \{\hat{\nu}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}\big)$ to the restricted linear program, we check for each i and t if

$$\max_{r_i \in \mathcal{R}_i, \boldsymbol{u}^i \in \mathcal{U}_i(r_i)} \sum_{j \ni i} p_{jt} u^i_j [\hat{\lambda}_{ijt} + \hat{\nu}_{i,t+1}(r_i - 1) - \hat{\nu}_{i,t+1}(r_i)] + \hat{\nu}_{i,t+1}(r_i) - \hat{\nu}_{it}(r_i)$$

is greater than zero. Note that the separation problem for (LR) is easy to solve since for $r_i \in \mathcal{R}_i \backslash \{0\}$, the maximum is attained by setting $u^i_j = \mathbb{1}_{[\hat{\lambda}_{ijt} + \hat{\nu}_{i,t+1}(r_i - 1) - \hat{\nu}_{i,t+1}(r_i) > 0]}$ for $j \ni i$. On the other hand, if $r_i = 0$, the only feasible solution is $u^i_j = 0$ for all $j \ni i$. We add violated constraints to (LR) and stop when we are within 1% of optimality.

Table 3 gives the objective function values of (PL) and (LR) and the CPU seconds when they are solved to within 1% of optimality. We solve the test problems using CPLEX 11.2 on a Pentium Core 2 Duo PC with 3 GHz CPU and 4 GB RAM. The first column gives the characteristics of the test problem in terms of (τ, N, α). The second column gives the objective function value of (PL), while the third column gives the CPU seconds required by (PL). The fourth and fifth columns do the same thing, but for (LR). Comparing the second and fourth columns, we see that the objective function values of (PL) and (LR) are very close; the differences are within 1%. On the other hand, the solution times for (PL) on the test problems with a relatively large number of spokes and time periods can be orders of magnitude greater than (LR). We note that it is possible to solve both (PL) and (LR) more efficiently; see [4] and [15]. Our goal here is to simply compare the objective function values and solution times of comparable implementations of both methods. However, our results are in broadly in line with the computational studies reported [4] and [15].

6 Conclusions

We make the following research contributions in this paper: (1) We show that the approximate dynamic programming approach with piecewise-linear basis functions (Farias and van Roy [4]) and the Lagrangian relaxation approach (Topaloglu [15]) are in fact equivalent. This result shows that there might be surprising connections between the approximate dynamic programming approach and the Lagrangian relaxation approach for complicated dynamic programs, and one can benefit from unifying forces as it were. (2) We show that the separation problem for the piecewise-linear approximation is solvable in polynomial-time. (3) We show that there exists a separable concave approximation that yields the tightest upper bound among all separable piecewise-linear approximations to the value function. This implies that the Lagrangian relaxation approach obtains the tightest upper bound among all separable piecewise-linear approximations that are upper bounds on the value function.

Problem	(PL)		(LR)	
(τ, N, α)	V^{PL}	CPU	V^{LR}	CPU
(25, 2, 1.0)	622	3	622	0.1
(25, 2, 1.2)	557	3	557	0.1
(25, 2, 1.6)	448	2	448	0.1
(25, 3, 1.0)	972	14	972	0.4
(25, 3, 1.2)	868	8	868	0.3
(25, 3, 1.6)	700	5	700	0.2
(25, 4, 1.0)	1,187	39	1,188	1
(25, 4, 1.2)	1,048	21	1,048	1
(25, 4, 1.6)	843	10	844	0.5
(50, 2, 1.0)	1,305	71	1,306	1
(50, 2, 1.2)	1,117	42	1,117	1
(50, 2, 1.6)	908	24	908	0.5
(50, 3, 1.0)	2,038	496	2,038	2
(50, 3, 1.2)	1,844	211	1,845	2
(50, 3, 1.6)	1,500	74	1,500	1
(50, 4, 1.0)	2,496	1,556	2,497	6
(50, 4, 1.2)	2,260	746	2,263	4
(50, 4, 1.6)	1,855	227	1,856	3
(100, 2, 1.0)	3,652	2,149	3,652	27
(100, 2, 1.2)	3,242	1,409	3,245	18
(100, 2, 1.6)	2,599	831	2,603	8
(100, 3, 1.0)	5,529	17,821	5,531	44
(100, 3, 1.2)	4,967	9,314	4,972	32
(100, 3, 1.6)	4,131	4,000	4,137	18
(100, 4, 1.0)	6,835	108,297	6,837	80
(100, 4, 1.2)	6,141	51,708	6,148	75
(100, 4, 1.6)	4,910	12,250	4,917	36

Table 3: Comparison of the upper bounds and solution times of (PL) and (LR) both solved to 1% of optimality by linear programming.

Our proof technique uses a novel minimal-binding-constraints argument. We give heuristic conditions for the minimal-binding-constraints technique to work. The checklist is based on a simple calculus-based argument which provides intuition and may be useful when applying a similar line of reasoning to other classes of loosely-coupled dynamic programs, in the framework of Hawkins [6]. Indeed, we sketch in the appendix how the equivalence result extends to network RM with overbooking under some assumptions on the denied-boarding cost function. We show that the result does not directly extend to choice-based network RM and discuss why—our insight is that the number of Lagrange multipliers plays a critical role in the equivalence, requiring problem-specific customization.

As to computational impact, we solved the piecewise-linear approximation using the linear-programming based separation we describe in this paper, but our results suggest that solving the Lagrangian relaxation is still faster. This is in line with the computational studies reported in Farias and van Roy [4], and not too surprising as solvability by separation is based on the ellipsoid algorithm that is well known to be slow in practice. Tong and Topaloglu [14] make a similar observation for the affine relaxation of Adelman [1]; they also find that the Lagrangian relaxation approach described in Kunnumkal and Topaloglu [10] is more efficient. Improving the efficiency of the separation, say by a faster, combinatorial, algorithm, would be an interesting area for future research.

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Appendix

Proof of Lemma 1:

Our analysis is essentially an adaptation of analogous structural results for the revenue management problem on a single resource (Talluri and van Ryzin [13]). We introduce some notation to simplify the expressions. Fixing a resource l, we let $\mathcal{R}_l(r_l) = \{\boldsymbol{x} \in \mathcal{R} \,|\, x_l = r_l\}$ be the set of capacity vectors where the capacity on resource l is fixed at r_l. Given a separable piecewise-linear approximation $\mathcal{V} = \{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$, we let

$$\begin{aligned}\epsilon_{lt}(r_l, \mathcal{V}) \quad &= \min_{\boldsymbol{r}\in\mathcal{R}_l(r_l), \boldsymbol{u}\in\mathcal{U}(\boldsymbol{r})} \Big\{ \sum_i v_{it}(r_i) - \sum_j p_{jt} u_j \big[f_j + \sum_{i\in j}[v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)]\big] \\ &\qquad - \sum_i v_{i,t+1}(r_i) \Big\}\end{aligned}$$

where the argument $\mathcal{V}$ emphasizes the dependence on the given approximation. Note that if $\mathcal{V}$ is feasible to (PL), then $\epsilon_{it}(r_i, \mathcal{V}) \geq 0$ for all t, i and $r_i \in \mathcal{R}_i$. We begin with a preliminary result.

Lemma 4. *There exists an optimal solution* $\hat{\mathcal{V}} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ *to* (PL) *such that for all* t, i *and* $r_i \in \mathcal{R}_i$, *we have* $\epsilon_{it}(r_i, \hat{\mathcal{V}}) = 0$.

Proof. Let $\mathcal{V} = \{v_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ be an optimal solution to problem (PL). Let s be the largest time index such that there exists a resource l and $r_l \in \mathcal{R}_l$ with $\epsilon_{ls}(r_l, \mathcal{V}) > 0$. Since $\mathcal{V}$ is feasible, this means that $\epsilon_{it}(r_i, \mathcal{V}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$. We consider decreasing $v_{ls}(r_l)$ alone by $\epsilon_{ls}(r_l, \mathcal{V})$ leaving all the other elements of $\mathcal{V}$ unchanged. That is, let $\hat{\mathcal{V}} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ where

$$\hat{v}_{it}(x) = \begin{cases} v_{it}(x) - \epsilon_{it}(x, v) & \text{if } i = l, t = s, x = r_l \\ v_{it}(x) & \text{otherwise.} \end{cases} \tag{20}$$

Note that since $\hat{v}_{it}(r_i) \leq v_{it}(r_i)$ for all t, i and $r_i \in \mathcal{R}_i$, we have $\sum_i \hat{v}_{i1}(r_i^1) \leq \sum_i v_{i1}(r_i^1)$. Next, we show that $\hat{\mathcal{V}}$ is feasible. Since $\hat{\mathcal{V}}$ differs from $\mathcal{V}$ only in one element, we only have to check those constraints where $\hat{v}_{ls}(r_l)$ appears. Observe that $\hat{v}_{ls}(r_l)$ appears only in the constraints for time periods $s-1$ and s. For time period $s-1$, we have

$$\begin{aligned}&\sum_j p_{j,s-1} u_j \big[f_j + \sum_{i\in j} \hat{v}_{is}(r_i - 1)\big] + \sum_i [1 - \sum_{j\ni i} p_{j,s-1} u_j] \hat{v}_{is}(r_i) \\ &\quad \leq \sum_j p_{j,s-1} u_j \big[f_j + \sum_{i\in j} v_{is}(r_i - 1)\big] + \sum_i [1 - \sum_{j\ni i} p_{j,s-1} u_j] v_{is}(r_i) \\ &\quad \leq \sum_i v_{i,s-1}(r_i) \\ &\quad = \sum_i \hat{v}_{i,s-1}(r_i)\end{aligned}$$

for all $\boldsymbol{r} \in \mathcal{R}$ and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})$, where the first inequality follows since $\hat{v}_{is}(r_i) \leq v_{is}(r_i)$ and $\sum_{j\ni i} p_{j,s-1} u_j \leq 1$, the second inequality follows from the feasibility of $\mathcal{V}$ and the equality follows from (20). For time period s, $\hat{v}_{ls}(r_l)$ appears only in constraints corresponding to $\boldsymbol{r} \in \mathcal{R}_l(r_l)$. For $\boldsymbol{r} \in \mathcal{R}_l(r_l)$, we have

$$\begin{aligned}
&\sum_i \hat{v}_{is}(r_i) \\
&\quad = \sum_i v_{is}(r_i) - \epsilon_{ls}(r_l, \mathcal{V}) \\
&\quad \geq \sum_j p_{js} u_j \big[f_j + \sum_{i \in j} \{ v_{i,s+1}(r_i - 1) - v_{i,s+1}(r_i) \} \big] + \sum_i v_{i,s+1}(r_i) \\
&\quad = \sum_j p_{js} u_j \big[f_j + \sum_{i \in j} \{ \hat{v}_{i,s+1}(r_i - 1) - \hat{v}_{i,s+1}(r_i) \} \big] + \sum_i \hat{v}_{i,s+1}(r_i)
\end{aligned}$$

for all $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{r})$, where the inequality follows from the definition of $\epsilon_{ls}(r_l, \mathcal{V})$ and the last equality follows from (20). Therefore $\hat{\mathcal{V}}$ is feasible, which implies that $\epsilon_{it}(r_i, \hat{\mathcal{V}}) \geq 0$ for all t, i and $r_i \in \mathcal{R}_i$. Next, we note from (20) that $\epsilon_{it}(r_i, \hat{\mathcal{V}}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$. For time period s, since $\hat{v}_{is}(r_i) \leq v_{is}(r_i)$ and $\hat{v}_{i,s+1}(r_i) = v_{i,s+1}(r_i)$, it follows that $\epsilon_{is}(r_i, \hat{\mathcal{V}}) \leq \epsilon_{is}(r_i, \mathcal{V})$. Therefore, if $\epsilon_{is}(r_i, \mathcal{V})$ was zero, then $\epsilon_{is}(r_i, \hat{\mathcal{V}})$ is also zero. Moreover, $\epsilon_{ls}(r_l, \hat{\mathcal{V}}) = 0 < \epsilon_{ls}(r_l, \mathcal{V})$.

To summarize, $\hat{\mathcal{V}}$ is an optimal solution with $\epsilon_{it}(r_i, \hat{\mathcal{V}}) = 0$ for all $t > s$, i and $r_i \in \mathcal{R}_i$ and $|\{\epsilon_{is}(r_i, \hat{\mathcal{V}}) \,|\, \epsilon_{is}(r_i, \hat{\mathcal{V}}) > 0\}| < |\{\epsilon_{is}(r_i, \mathcal{V}) \,|\, \epsilon_{is}(r_i, \mathcal{V}) > 0\}|$. We repeat the above procedure finitely many times to obtain an optimal solution $\hat{\mathcal{V}}$ with $\epsilon_{it}(r_i, \hat{\mathcal{V}}) = 0$ for all $t \geq s$, i and $r_i \in \mathcal{R}_i$. Repeating the entire procedure for time periods $s-1, \ldots, 1$ completes the proof. □

We are ready to prove Lemma 1. By Lemma 4, we can pick an optimal solution $\hat{\mathcal{V}} = \{\hat{v}_{it}(r_i) \,|\, \forall t, i, r_i \in \mathcal{R}_i\}$ such that $\epsilon_{it}(r_i, \hat{\mathcal{V}}) = 0$ for all t, i and $r_i \in \mathcal{R}_i$. The proof proceeds by induction on the time periods. It is easy to see that the result holds for time period τ. Fix a resource l and assume that statements (i) and (ii) of the lemma hold for all time periods $s > t$. We show below that statements (i) and (ii) hold for time period t as well.

Since $\hat{v}_{it}(-1) = -\infty$, statement (i) holds trivially for $r_l = 0$. For $r_l = 1$, Lemma 4 implies that there exists $\boldsymbol{x} \in \mathcal{R}_l(0)$ and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{x})$ such that

$$\begin{aligned}
\hat{v}_{lt}(0) + \sum_{i \neq l} \hat{v}_{it}(x_i) = \sum_{j \not\ni l} p_{jt} u_j [f_j &+ \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)]] \\
&+ \hat{v}_{l,t+1}(0) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i).
\end{aligned} \tag{21}$$

where $\mathbb{1}_{[\cdot]}$ denotes the indicator function and we use the fact that since $x_l = 0$, $u_j = 0$ for all $j \ni l$. Next, consider the capacity vector $\boldsymbol{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l = 1$. Since $\boldsymbol{x} \leq \boldsymbol{y}$, $\mathcal{U}(\boldsymbol{x}) \subset \mathcal{U}(\boldsymbol{y})$ and it follows that $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{y})$. Since $\hat{\mathcal{V}}$ is feasible, we have

$$\begin{aligned}
\hat{v}_{lt}(1) + \sum_{i \neq l} \hat{v}_{it}(x_i) \geq \sum_{j \not\ni l} p_{jt} u_j [f_j &+ \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)]] \\
&+ \hat{v}_{l,t+1}(1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i).
\end{aligned} \tag{22}$$

Subtracting (21) from (22), we have $\hat{v}_{lt}(1) - \hat{v}_{lt}(0) \geq \hat{v}_{l,t+1}(1) - \hat{v}_{l,t+1}(0)$.

We next show that statement (i) holds for $r_l \in \mathcal{R}_l \backslash \{0, 1\}$. By Lemma 4, there exists $\boldsymbol{x} \in \mathcal{R}_l(r_l - 1)$

and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{x})$ such that

$$\begin{aligned}
&\hat{v}_{lt}(r_l - 1) + \sum_{i \neq l} \hat{v}_{it}(x_i) \\
&\quad = \sum_j p_{jt} u_j [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l \in j]} [\hat{v}_{l,t+1}(r_l - 2) - \hat{v}_{l,t+1}(r_l - 1)]] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l - 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (23)
\end{aligned}$$

Now, consider the capacity vector $\boldsymbol{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since $\boldsymbol{x} \leq \boldsymbol{y}$, $\mathcal{U}(\boldsymbol{x}) \subset \mathcal{U}(\boldsymbol{y})$ and it follows that $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{y})$. Since $\hat{\mathcal{V}}$ is feasible, we have

$$\begin{aligned}
&\hat{v}_{lt}(r_l) + \sum_{i \neq l} \hat{v}_{it}(x_i) \\
&\quad \geq \sum_j p_{jt} u_j [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l \in j]} [\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l)]] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (24)
\end{aligned}$$

Subtracting (23) from (24), we get

$$\begin{aligned}
&\hat{v}_{lt}(r_l) - \hat{v}_{lt}(r_l - 1) \\
&\quad \geq \sum_j p_{jt} u_j \mathbb{1}_{[l \in j]} [2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2)] + \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1) \\
&\quad \geq \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1). \qquad (25)
\end{aligned}$$

Note that the last inequality follows, since by induction assumption (ii), we have

$$2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2) \geq 0.$$

Next, we show that statement (ii) holds for time period t. Since $\hat{v}_{it}(-1) = -\infty$, statement (ii) holds trivially for $r_l = 0$. For $r_l \in \mathcal{R}_l \backslash \{0, r_l^1\}$, Lemma 4 implies that there exists $\boldsymbol{x} \in \mathcal{R}_l(r_l + 1)$ and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{x})$ such that

$$\begin{aligned}
&\hat{v}_{lt}(r_l + 1) + \sum_{i \neq l} \hat{v}_{it}(x_i) \\
&\quad = \sum_j p_{jt} u_j [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l \in j]} [\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l + 1)]] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l + 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (26)
\end{aligned}$$

Now consider the capacity vector $\boldsymbol{y}$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since $r_l \geq 1$, $u_j \leq r_l$ for all $j \ni l$. Since $y_i = x_i$ for $i \neq l$ and $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{x})$, we have that $u_j \leq y_i$ for all $j \ni i$. That is, we have $\boldsymbol{u} \in \mathcal{U}(\boldsymbol{y})$. Since $\hat{\mathcal{V}}$ is feasible, we have

$$\begin{aligned}
&\hat{v}_{lt}(r_l) + \sum_{i \neq l} \hat{v}_{it}(x_i) \\
&\quad \geq \sum_j p_{jt} u_j [f_j + \sum_{i \neq l} \mathbb{1}_{[i \in j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[l \in j]} [\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l)]] \\
&\qquad\qquad + \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \qquad (27)
\end{aligned}$$

Subtracting (27) from (26), we get

$$\begin{aligned}
&\hat{v}_{lt}(r_l+1)-\hat{v}_{lt}(r_l)\\
&\quad\le \sum_j p_{jt}u_j \mathbb{1}_{[l\in j]}[2\hat{v}_{l,t+1}(r_l)-\hat{v}_{l,t+1}(r_l+1)-\hat{v}_{l,t+1}(r_l-1)]+\hat{v}_{l,t+1}(r_l+1)-\hat{v}_{l,t+1}(r_l)\\
&\quad\le 2\hat{v}_{l,t+1}(r_l)-\hat{v}_{l,t+1}(r_l+1)-\hat{v}_{l,t+1}(r_l-1)+\hat{v}_{l,t+1}(r_l+1)-\hat{v}_{l,t+1}(r_l)\\
&\quad= \hat{v}_{l,t+1}(r_l)-\hat{v}_{l,t+1}(r_l-1)\\
&\quad\le \hat{v}_{lt}(r_l)-\hat{v}_{lt}(r_l-1).
\end{aligned}$$

Note that the second inequality above follows, since by induction assumption (ii), $2\hat{v}_{l,t+1}(r_l)-\hat{v}_{l,t+1}(r_l+1)-\hat{v}_{l,t+1}(r_l-1)\ge 0$ and $\sum_j p_{jt}u_j\mathbb{1}_{[l\in j]}\le 1$ and the last inequality follows from (25). Finally, for $r_l=r_l^1$, following a similar analysis, we get

$$\begin{aligned}
\hat{v}_{lt}(r_l^1)-\hat{v}_{lt}(r_l^1-1)\ge &\sum_j p_{jt}u_j\mathbb{1}_{[l\in j]}[2\hat{v}_{l,t+1}(r_l^1-1)-\hat{v}_{l,t+1}(r_l^1)-\hat{v}_{l,t+1}(r_l^1-2)]\\
&+\hat{v}_{l,t+1}(r_l^1)-\hat{v}_{l,t+1}(r_l^1-1).
\end{aligned}$$

By induction assumption (ii), $2\hat{v}_{l,t+1}(r_l^1-1)-\hat{v}_{l,t+1}(r_l^1)-\hat{v}_{l,t+1}(r_l^1-2)\ge 0$ and $\hat{v}_{l,t+1}(r_l^1)-\hat{v}_{l,t+1}(r_l^1-1)\ge \hat{v}_{l,t+1}(r_l^1+1)-\hat{v}_{l,t+1}(r_l^1)$. We have

$$\hat{v}_{lt}(r_l^1)-\hat{v}_{lt}(r_l^1-1)\ge \hat{v}_{l,t+1}(r_l^1+1)-\hat{v}_{l,t+1}(r_l^1)=0=\hat{v}_{lt}(r_l^1+1)-\hat{v}_{lt}(r_l^1).$$

Therefore, statements (i) and (ii) hold at time period t for resource l. This completes the proof since resource l was an arbitrary choice. □

Proof of Proposition 1:

Suppose for product j and time t, we have $\sum_{i'\in j}\lambda_{i',j,t}>f_j$. This implies that for some $i\in j$, $\lambda_{ijt}>0$. Let $\delta=\min\{\lambda_{ijt},\sum_{i'\in j}\lambda_{i',j,t}-f_j\}>0$ and let $\{\hat{\lambda}_{i',j',t'}\,|\,\forall t',j',i'\in j'\}$ be the same as $\{\lambda_{i',j',t'}\,|\,\forall t',j',i'\in j'\}$ except that $\hat{\lambda}_{ijt}=\lambda_{ijt}-\delta<\lambda_{ijt}$.

Note that $\vartheta_{i1}^{\hat{\lambda}}(r_i^1)\le\vartheta_{i1}^{\lambda}(r_i^1)$ as we have reduced the revenue associated with product j at time period t, keeping all other product revenues the same. As the first part of the right hand side of (3) is unaffected by this change, we get

$$V_1^{\hat{\lambda}}(\boldsymbol{r}^1)\le V_1^{\lambda}(\boldsymbol{r}^1).$$

If after performing this step, we still have $\sum_{i'\in j}\hat{\lambda}_{i',j,t}>f_j$ for some product j and time t, we can repeat this step for another resource $i\in j$ until we have $\sum_{i'\in j}\lambda_{i',j,t}\le f_j$ for all j and t.

Now suppose for some product j and time t, $\sum_{i'\in j}\lambda_{i',j,t}<f_j$. Fix $i\in j$ and let $\{\hat{\lambda}_{i',j',t'}\,|\,\forall t',j',i'\in j'\}$ be the same as $\{\lambda_{i',j',t'}\,|\,\forall t',j',i'\in j'\}$ except that $\hat{\lambda}_{ijt}=\lambda_{ijt}+\delta$, where $\delta=f_j-\sum_{i\in j}\lambda_{ijt}>0$.

We have

$$\vartheta_{i1}^{\hat{\lambda}}(r_i^1)\le\vartheta_{i1}^{\lambda}(r_i^1)+\delta.$$

This is because, by increasing the revenue associated with product j at time t by δ, while keeping all other product revenues the same, we cannot increase the optimal expected revenue from resource

i by more than δ. However,

$$[f_j - \sum_{i' \in j} \hat{\lambda}_{i',j,t}]^+ = [f_j - \sum_{i' \in j} \lambda_{i',j,t}]^+ - \delta$$

and so

$$V_1^{\hat{\lambda}}(\boldsymbol{r}^1) \leq V_1^{\lambda}(\boldsymbol{r}^1).$$

By repeating this step, if necessary, we obtain an optimal solution that satisfies $\sum_{i' \in j} \lambda_{i',j,t} = f_j$ for all j and t.

Finally, we show that there exists optimal Lagrange multipliers $\lambda_{i',j',t'} \geq 0$ for all t', j' and $i' \in j'$. Suppose $\lambda_{ijt} < 0$ for some product j, resource $i \in j$ and time t. Since the Lagrange multipliers sum up to f_j, there exists a resource $l \in j$ such that $\lambda_{ljt} > 0$. Let $\delta = \min\{[\lambda_{ijt}]^+, \lambda_{ljt}\} > 0$ and let $\{\hat{\lambda}_{i',j',t'} \,|\, \forall t', j', i' \in j'\}$ be the same as $\{\lambda_{i',j',t'} \,|\, \forall t', j', i' \in j'\}$ except that $\hat{\lambda}_{ijt} = \lambda_{ijt} + \delta$ and $\hat{\lambda}_{ljt} = \lambda_{ljt} - \delta$. Observe that $\vartheta_{i1}^{\hat{\lambda}}(r_i^1) \leq \vartheta_{i1}^{\lambda}(r_i^1)$ and $\vartheta_{l1}^{\hat{\lambda}}(r_l^1) \leq \vartheta_{l1}^{\lambda}(r_l^1)$ so that $V_1^{\hat{\lambda}}(\boldsymbol{r}^1) \leq V_1^{\lambda}(\boldsymbol{r}^1)$. If there is still some Lagrange multiplier that is negative, we repeat the step until we have $\lambda_{i',j',t'} \geq 0$ for all t', j' and $i' \in j'$. □

Proof of Lemma 3:

Note that an optimal solution to problem (7) satisfies $u_j^i = \mathbb{1}_{[\lambda_{ijt} \geq \psi_{i,t+1}(r_i)]}$ for all $j \ni i$, where we use the fact that $\psi_{i,t+1}(0) = \infty$. Therefore, using the convention that $0 \times -\infty = 0$, problem (7) can be written as $\max_{r \in \mathcal{R}_i} \sum_{j \ni i} p_{jt} \mathbb{1}_{[\lambda_{ijt} \geq \psi_{i,t+1}(r)]} [\lambda_{ijt} - \psi_{i,t+1}(r)] + \Delta_{it}(r)$. On the other hand, there exists an optimal solution $\big(\hat{w}_{it}, \{\hat{z}_{ijtr} \,|\, \forall j \ni i, r \in \mathcal{R}_i\}\big)$ to $(SepLR_i)$ such that $\hat{z}_{ijtr} = p_{jt}[\lambda_{ijt} - \psi_{i,t+1}(r)]^+ = p_{jt} \mathbb{1}_{[\lambda_{ijt} \geq \psi_{i,t+1}(r)]} [\lambda_{ijt} - \psi_{i,t+1}(r)]$ for all $j \ni i$ and $r \in \mathcal{R}_i$. Moreover, there exists an $\hat{r} \in \mathcal{R}_i$ such that $\hat{w}_{it} = \sum_{j \ni i} \hat{z}_{ijt\hat{r}} + \Delta_{it}(\hat{r})$. Therefore, $\hat{w}_{it} = \max_{r \in \mathcal{R}_i} \sum_{j \ni i} \hat{z}_{ijtr} + \Delta_{it}(r)$. □

Proof of Proposition 4:

Choose $(\hat{\lambda}, \hat{w}, \hat{z})$ to be an optimal solution to $(SepLR)$ with a minimal set $\bigcup_{i' \in \mathcal{I}} B_{i'}(\hat{\lambda}, \hat{w}, \hat{z})$. So, there is no other optimal solution $(\hat{\lambda}', \hat{w}', \hat{z}')$ which has a set of binding constraints that is a strict subset of the binding constraints of $(\hat{\lambda}, \hat{w}, \hat{z})$; that is,

$$\bigcup_{i' \in \mathcal{I}} B_{i'}(\hat{\lambda}', \hat{w}', \hat{z}') \subsetneq \bigcup_{i' \in \mathcal{I}} B_{i'}(\hat{\lambda}, \hat{w}, \hat{z}).$$

For resource i', we let $\hat{r}_{i'} = \max\{r \,|\, r \in B_{i'}(\hat{\lambda}, \hat{w}, \hat{z})\}$, so that for all $r > \hat{r}_{i'}$, $\xi_{i',t}(r) > 0$. Now suppose there exists a product j such that for $i \in j$ we have $\hat{\lambda}_{ijt} < \psi_{i,t+1}(\hat{r}_i)$, while for $l \in j$, we have $\hat{\lambda}_{ljt} > \psi_{l,t+1}(\hat{r}_l)$. In this case, we construct an optimal solution $(\bar{\lambda}, \bar{w}, \bar{z})$ with $\bigcup_{i' \in \mathcal{I}} B_{i'}(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq \bigcup_{i' \in \mathcal{I}} B_{i'}(\hat{\lambda}, \hat{w}, \hat{z})$, which contradicts $(\hat{\lambda}, \hat{w}, \hat{z})$ being an optimal solution with a minimal set of binding constraints of type (13) amongst all optimal solutions.

Recall (from Lemma 1) that $\bar{v}_{i',t}(r_{i'}) - \bar{v}_{i',t}(r_{i'} - 1) \geq \bar{v}_{i',t}(r_{i'} + 1) - \bar{v}_{i',t}(r_{i'})$ for all i', $r_{i'} \in \mathcal{R}_{i'}$. Thus, we have

$$\psi_{i',t+1}(r_{i'}) \geq \psi_{i',t+1}(r_{i'} + 1) \geq 0, \quad \forall r_{i'} \in \mathcal{R}_{i'} \tag{28}$$

(≥ 0 as we set $\bar{v}_{i',t}(r_{i'}^1 + 1) = \bar{v}_{i',t}(r_{i'}^1)$; cf. Lemma 1). Also recall that we had assumed that $p_{jt} > 0$ without loss of generality.

Let

$$\epsilon = \min\left\{\psi_{i,t+1}(\hat{r}_i) - \hat{\lambda}_{ijt}, \hat{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l), \min\{\xi_{it}(r) \,|\, r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})\}\right\} > 0, \tag{29}$$

with the understanding that if $B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$ is empty, then $\min\{\xi_{it}(r) \,|\, r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})\} = \infty$.

We construct a solution $(\bar{\lambda}, \bar{w}, \bar{z})$ in the following manner. Pick $\delta \in (0, \epsilon)$ and let

$$\bar{\lambda}_{i',k,t} = \begin{cases} \hat{\lambda}_{ijt} + \delta & \text{if } i' = i, k = j \\ \hat{\lambda}_{ljt} - \delta & \text{if } i' = l, k = j \\ \hat{\lambda}_{i',k,t} & \text{otherwise,} \end{cases}$$

$\bar{w}_{i',t} = \hat{w}_{i',t}$ for all i' and

$$\bar{z}_{i',k,t,r} = \begin{cases} p_{jt}[\bar{\lambda}_{ijt} - \psi_{i,t+1}(r)]^+ & \text{if } i' = i, k = j, r \in B_i^c \\ p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+ & \text{if } i' = l, k = j, r \in \mathcal{R}_l \\ \hat{z}_{i',k,t,r} & \text{otherwise.} \end{cases}$$

Note that by construction, $(\bar{\lambda}, \bar{w}, \bar{z})$ has the same elements as $(\hat{\lambda}, \hat{w}, \hat{z})$ except that $\bar{\lambda}_{ijt} = \hat{\lambda}_{ijt} + \delta$, $\bar{\lambda}_{ljt} = \hat{\lambda}_{ljt} - \delta$, $\bar{z}_{ijtr} = p_{jt}[\bar{\lambda}_{ijt} - \psi_{i,t+1}(r)]^+$ for $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$ and $\bar{z}_{ljtr} = p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+$ for all $r \in \mathcal{R}_l$. We begin with some preliminary results.

Lemma 5. *(i) The solution $(\bar{\lambda}, \bar{w}, \bar{z})$ satisfies constraints (13), (14) and (17) for all $i' \notin \{i, l\}$.*
(ii) $B_{i'}(\bar{\lambda}, \bar{w}, \bar{z}) = B_{i'}(\hat{\lambda}, \hat{w}, \hat{z})$ for all $i' \notin \{i, l\}$.

Proof. The proof follows by noting that $(\hat{\lambda}, \hat{w}, \hat{z})$ is feasible and $\bar{w}_{i',t} = \hat{w}_{i',t}$, $\bar{\lambda}_{i',k,t} = \hat{\lambda}_{i',k,t}$ and $\bar{z}_{i',k,t,r} = \hat{z}_{i',k,t,r}$ for all $i' \notin \{i, l\}, r \in \mathcal{R}_{i'}$ and $k \ni i'$. □

Lemma 6. *(i) The solution $(\bar{\lambda}, \bar{w}, \bar{z})$ satisfies constraints (13), (14) and (17) for resource i.*
(ii) $B_i(\bar{\lambda}, \bar{w}, \bar{z}) = B_i(\hat{\lambda}, \hat{w}, \hat{z})$.

Proof. We first consider constraints (13). For $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$, we have

$$\bar{z}_{ijtr} = p_{jt}[\hat{\lambda}_{ijt} + \delta - \psi_{i,t+1}(r)]^+ \le p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(r)]^+ + p_{jt}\delta < \hat{z}_{ijtr} + \epsilon \le \hat{z}_{ijtr} + \xi_{it}(r), \tag{30}$$

where the second inequality uses $\delta < \epsilon$, and the fact that since $(\hat{\lambda}, \hat{w}, \hat{z})$ is feasible, $\hat{z}_{ijtr} \ge p_{jt}[\hat{\lambda}_{ijt} - \psi_{i,t+1}(r)]$ and $\hat{z}_{ijtr} \ge 0$. Therefore, for all $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$, $\bar{w}_{it} = \hat{w}_{it} = \sum_{k \ni i, k \neq j} \hat{z}_{iktr} + \hat{z}_{ijtr} + \Delta_{it}(r) + \xi_{it}(r) > \sum_{k \ni i} \bar{z}_{iktr} + \Delta_{it}(r)$, where the second equality follows by definition of $\xi_{it}(r)$ and the inequality uses (30) and the fact that $\bar{z}_{iktr} = \hat{z}_{iktr}$ for all $k \neq j$. Note also that the constraint is nonbinding for $(\bar{\lambda}, \bar{w}, \bar{z})$. Since $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$, the constraint was nonbinding for $(\hat{\lambda}, \hat{w}, \hat{z})$ as well. On the other hand, for $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$ since we have $\bar{z}_{ijtr} = \hat{z}_{ijtr}$ and $\bar{w}_{it} = \hat{w}_{it}$, $(\bar{\lambda}, \bar{w}, \bar{z})$ satisfies constraint (13) as an equality. Therefore, constraints (13) are binding for $(\bar{\lambda}, \bar{w}, \bar{z})$. Since $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$, these constraints were binding for $(\hat{\lambda}, \hat{w}, \hat{z})$ as well. Therefore, $B_i(\bar{\lambda}, \bar{w}, \bar{z}) = B_i(\hat{\lambda}, \hat{w}, \hat{z})$, which proves part (ii) of the lemma.

To complete the proof of part(i), we verify that constraints (14) and (17) are satisfied by $(\bar{\lambda}, \bar{w}, \bar{z})$. For $r \in B_i^c(\hat{\lambda}, \hat{w}, \hat{z})$, $\bar{z}_{ijtr} = p_{jt}[\bar{\lambda}_{ijt} - \psi_{i,t+1}(r)]^+$ satisfies constraints (14) and (17). For $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$, since $\hat{r}_i = \max\{r \,|\, r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})\}$, we have $\psi_{i,t+1}(r) \ge \psi_{i,t+1}(\hat{r}_i)$ (from 28). Constraint (14) is satisfied since

$$\begin{aligned} p_{jt}[\bar{\lambda}_{ijt} - \psi_{i,t+1}(r)] = p_{jt}[\hat{\lambda}_{ijt} + \delta - \psi_{i,t+1}(r)] \le p_{jt}[\hat{\lambda}_{ijt} + \delta - \psi_{i,t+1}(\hat{r}_i)] < p_{jt}[\hat{\lambda}_{ijt} + \epsilon - \psi_{i,t+1}(\hat{r}_i)] \\ \le 0 \le \hat{z}_{ijtr} = \bar{z}_{ijtr}, \end{aligned}$$

where the penultimate inequality follows from (29). Constraint (17) is satisfied since for $r \in B_i(\hat{\lambda}, \hat{w}, \hat{z})$, $\bar{z}_{ijtr} = \hat{z}_{ijtr} \geq 0$. ☐

Lemma 7. *(i) The solution $(\bar{\lambda}, \bar{w}, \bar{z})$ satisfies constraints (13), (14) and (17) for resource l.*
(ii) $B_l(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq B_l(\hat{\lambda}, \hat{w}, \hat{z})$.

Proof. We first consider constraints (13). By definition, $\bar{z}_{ljtr} = p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+$ for all $r \in \mathcal{R}_l$. On the other hand, since $(\hat{\lambda}, \hat{w}, \hat{z})$ is a feasible solution, $\hat{z}_{ljtr} \geq p_{jt}[\hat{\lambda}_{ljt} - \psi_{i,t+1}(r)]^+$ for all $r \in \mathcal{R}_l$. Since $\hat{\lambda}_{ljt} > \bar{\lambda}_{ljt}$, we have $\hat{z}_{ljtr} \geq \bar{z}_{ljtr}$ for all $r \in \mathcal{R}_l$. Using the fact that $\bar{z}_{lktr} = \hat{z}_{lktr}$ for all $k \neq j$ and $r \in \mathcal{R}_l$, we have

$$\bar{w}_{lt} = \hat{w}_{lt} \geq \sum_{k \in l} \hat{z}_{lktr} + \Delta_{lt}(r) = \sum_{k \in l} \bar{z}_{lktr} + \Delta_{lt}(r)$$

for all $r \in \mathcal{R}_l$. Therefore $(\bar{\lambda}, \bar{w}, \bar{z})$ satisfies constraints (13).

Next, we look at the number of binding constraints of type (13). Since $\bar{w}_{lt} = \hat{w}_{lt}$ and $\bar{z}_{lktr} \leq \hat{z}_{lktr}$ for all k and $r \in \mathcal{R}_l$, $B_l(\bar{\lambda}, \bar{w}, \bar{z}) \subseteq B_l(\hat{\lambda}, \hat{w}, \hat{z})$. By definition $\hat{r}_l \in B_l(\hat{\lambda}, \hat{w}, \hat{z})$. We show that $\hat{r}_l \notin B_l(\bar{\lambda}, \bar{w}, \bar{z})$, which proves part (ii) of the lemma. First, note that

$$\bar{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l) = \hat{\lambda}_{ljt} - \delta - \psi_{l,t+1}(\hat{r}_l) > \hat{\lambda}_{ljt} - \epsilon - \psi_{l,t+1}(\hat{r}_l) \geq 0, \tag{31}$$

where the last inequality uses (29). Therefore,

$$\begin{aligned} \bar{z}_{l,j,t,\hat{r}_l} = p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l)]^+ = p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l)] &< p_{jt}[\hat{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l)] \\ &\leq p_{jt}[\hat{\lambda}_{ljt} - \psi_{l,t+1}(\hat{r}_l)]^+ \leq \hat{z}_{l,j,t,\hat{r}_l}, \end{aligned}$$

where the second equality follows from (31), the first inequality holds since $\bar{\lambda}_{ljt} < \hat{\lambda}_{ljt}$ and the last inequality holds since $(\hat{\lambda}, \hat{w}, \hat{z})$ is a feasible solution. The above chain of inequalities imply that $\bar{z}_{l,j,t,\hat{r}_l} < \hat{z}_{l,j,t,\hat{r}_l}$. Therefore, we have

$$\bar{w}_{lt} = \hat{w}_{lt} = \sum_{k \ni l, k \neq j} \hat{z}_{l,k,t,\hat{r}_l} + \hat{z}_{l,j,t,\hat{r}_l} + \Delta_{lt}(\hat{r}_l) > \sum_{k \ni l, k \neq j} \bar{z}_{l,k,t,\hat{r}_l} + \bar{z}_{l,j,t,\hat{r}_l} + \Delta_{lt}(\hat{r}_l).$$

Consequently, $\hat{r}_l \notin B_l(\bar{\lambda}, \bar{w}, \bar{z})$ and $B_l(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq B_l(\hat{\lambda}, \hat{w}, \hat{z})$.

To complete the proof of part(i), we verify that constraints (14) and (17) are satisfied by $(\bar{\lambda}, \bar{w}, \bar{z})$. This is trivial, since by definition we have $\bar{z}_{ljtr} = p_{jt}[\bar{\lambda}_{ljt} - \psi_{l,t+1}(r)]^+$. ☐

We are ready to prove Proposition 4. By construction $(\bar{\lambda}, \bar{w}, \bar{z})$ satisfies constraints (15). Since $\bar{\lambda}_{ljt} > \hat{\lambda}_{ljt} - \epsilon \geq \psi_{l,t+1}(\hat{r}_l) \geq 0$ (from 28), $(\bar{\lambda}, \bar{w}, \bar{z})$ also satisfies constraints (16). This together with parts (i) of Lemmas 5, 6 and 7 imply that $(\bar{\lambda}, \bar{w}, \bar{z})$ is a feasible solution. Since $\bar{w}_{i',t} = \hat{w}_{i',t}$ for all i', $(\bar{\lambda}, \bar{w}, \bar{z})$ is also optimal. Parts (ii) of Lemmas 5, 6 and 7 together imply that

$$\bigcup_{i' \in \mathcal{I}} B_{i'}(\bar{\lambda}, \bar{w}, \bar{z}) \subsetneq \bigcup_{i' \in \mathcal{I}} B_{i'}(\hat{\lambda}, \hat{w}, \hat{z}).$$

That is, $(\bar{\lambda}, \bar{w}, \bar{z})$ is an optimal solution whose set of binding constraints is a strict subset of those of $(\hat{\lambda}, \hat{w}, \hat{z})$, and we get a contradiction to our minimality assumption. ☐

Extension to an overbooking model:

In this section we sketch how the results can be extended to a network RM model with overbooking studied in Karaesmen and van Ryzin [7] and Erdelyi and Topaloglu [3].

Let x_j denote the number of sales of product j and $\boldsymbol{x} = [x_j]$ be the vector of the number of sales. Let $\tilde{V}_{\tau+1}(\boldsymbol{x})$ denote the total revenue including the denied boarding penalty cost when we have $\boldsymbol{x}$ reservations at the end of the booking period. The optimality equation is

$$\tilde{V}_t(\boldsymbol{x}) = \max_{\boldsymbol{u} \in \{0,1\}^n} \sum_j p_{jt} u_j \left[f_j + \tilde{V}_{t+1}(\boldsymbol{x} + \boldsymbol{e}^j) - \tilde{V}_{t+1}(\boldsymbol{x}) \right] + \tilde{V}_{t+1}(\boldsymbol{x}).$$

where $\boldsymbol{e}^j$ is an n-dimensional 0-1 vector with a 1 at the jth position and 0 elsewhere.

We make the following assumption on the terminal revenue (the denied-boarding cost): $\tilde{V}_{\tau+1}(\boldsymbol{x}) = \hat{V}_{\tau+1}(A\boldsymbol{x})$, where A is the resource-product incidence matrix, the (i,j)th element of which indicates if product j consumes resource i. That is, the denied boarding cost is not a function of the number of reservations; it only depends on the total capacity used up on each resource. This assumption is primarily driven by tractability considerations and used also, for example, in Karaesmen and van Ryzin [7] and Erdelyi and Topaloglu [3].

Let $\boldsymbol{y} = A\boldsymbol{x}$ and note that y_i gives the total capacity consumed on resource i, when the sales vector is $\boldsymbol{x}$. With the above assumption on the terminal revenue, the optimality equation can be equivalently written as

$$\hat{V}_t(\boldsymbol{y}) = \max_{\boldsymbol{u} \in \{0,1\}^n} \sum_j p_{jt} u_j \left[f_j + \hat{V}_{t+1}(\boldsymbol{y} + \sum_{i \in j} \boldsymbol{e}^i) - \hat{V}_{t+1}(\boldsymbol{y}) \right] + \hat{V}_{t+1}(\boldsymbol{y}),$$

where $\boldsymbol{e}^i$ is an m-dimensional 0-1 vector with a 1 at the ith position and 0 elsewhere.

We consider the following piecewise-linear approximation: $\hat{V}_t(\boldsymbol{y}) \approx \sum_i \hat{v}_{it}(y_i)$, where we *impose* a restriction that $\hat{v}_{it}(\cdot)$ are discrete-concave functions (but not necessarily increasing). Also, let c_i be a large enough number that the denied-boarding costs are prohibitively high (equivalently, revenue losses are huge) if the capacity consumption on resource i exceeds c_i. That is, $\hat{v}_{i\tau}(c_i)$ is finite but $\hat{v}_{i\tau}(c_i + 1)$ is (assumed to be) $-\infty$. Letting $\mathcal{Y}_i = \{0, \ldots, c_i\}$ and $\mathcal{Y} = \prod_i \mathcal{Y}_i$, the linear program associated with the piecewise-linear approximation is

$$
\begin{aligned}
\hat{V}^{OPL} = \min_{\hat{v}} \quad & \textstyle\sum_i \hat{v}_{i1}(0) \\
\text{s.t} \quad & \\
(OPL) \qquad & \textstyle\sum_i \hat{v}_{it}(y_i) \geq \sum_j p_{jt} u_j \left[f_j + \sum_{i \in j} \{ \hat{v}_{i,t+1}(y_i + 1) - \hat{v}_{i,t+1}(y_i) \} \right] \\
& \qquad + \sum_i \hat{v}_{i,t+1}(y_i) \quad \forall t, \boldsymbol{y} \in \mathcal{Y}, \boldsymbol{u} \in \{0,1\}^n \\
& \hat{v}_{it}(y_i) - \hat{v}_{it}(y_i - 1) \leq \hat{v}_{it}(y_i - 1) - \hat{v}_{it}(y_i - 2) \quad \forall t, i, y_i \in \mathcal{Y}_i.
\end{aligned}
$$

where the objective function reflects the fact that we have no reservations and hence no capacity consumption at the start of the booking period.

Let $r_i = c_i - y_i$ denote the remaining capacity on resource i. Note that since $y_i \in \mathcal{Y}_i$, we have $r_i \in \mathcal{Y}_i$ as well. Let $v_{it}(r_i) = \hat{v}_{it}(c_i - r_i) = \hat{v}_{it}(y_i)$. Since $\hat{v}_{it}(\cdot)$ is concave, $v_{it}(\cdot)$ is also concave. Therefore, $v_{it}(r_i) - v_{it}(r_i - 1)$, the marginal value of capacity on resource i, is nonincreasing. Making the change of variables, linear program (OPL) can be equivalently written as

$$(OPL_{eq}) \quad \begin{aligned} V^{OPL} = \min_{v} \quad & \textstyle\sum_i v_{i1}(c_i) \\ \text{s.t} \quad & \\ & \textstyle\sum_i v_{it}(r_i) \geq \sum_j p_{jt} u_j \big[f_j + \sum_{i \in j} \{ v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i) \} \big] \\ & \qquad + \sum_i v_{i,t+1}(r_i) \quad \forall t, \boldsymbol{r} \in \mathcal{Y}, \boldsymbol{u} \in \{0,1\}^n \\ & v_{it}(r_i) - v_{it}(r_i - 1) \leq v_{it}(r_i - 1) - v_{it}(r_i - 2) \quad \forall t, i, r_i \in \mathcal{Y}_i. \end{aligned}$$

Linear program (OPL_{eq}) is almost similar to (PL), the linear program associated the piecewise-linear approximation for the network revenue management problem where all reservations show up, except that the concavity conditions need not be redundant and the acceptance decisions are not constrained by the remaining capacities on the resources. However, notice that we still have 1) a threshold type optimal control, since $u_j = 1$ only if f_j exceeds $\sum_{i \in j} v_{i,t+1}(r_i) - v_{i,t+1}(r_i - 1)$ and 2) the marginal value of capacity, $v_{i,t+1}(r_i) - v_{i,t+1}(r_i - 1)$, is nonincreasing by the concavity of $v_{i,t+1}(\cdot)$. The minimal-binding-constraints argument can be used to show that the separation problem decomposes by resource.