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**A Comparison of Polarization Measures**

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# A Comparison of Polarization Measures

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## Abstract

This paper provides a systematic classification of the different measures of polarization based on their properties. Together with the axioms proposed in Duclos, Esteban and Ray (2004) and in Wang and Tsui (2000) we consider three additional properties. We examine which properties are common to all indices and which set them apart.

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# 1 Introduction

Over the past years there has been an increasing interest in the notion and measurement of polarization.<sup>1</sup> This interest seems to have been triggered by the emergence of social phenomena that could not be properly captured by the more traditional measures of inequality.<sup>2</sup> The apparent shrinking of the middle class in some western societies and the frequency and amplitude of social conflicts, both appear in the literature as motivating evidence.

A number of polarization indices have been proposed. The purpose of this paper is to provide a systematic framework against which to locate these alternative polarization measures.

The term "polarization" is of fairly recent use in Economics, but has a long tradition in Political Science. Yet, its precise meaning has remained somewhat ambiguous. In a very broad sense, there is agreement that polarization is designed to capture the appearance (or disappearance) of groups in a distribution. But, agreement ends here. One family of measures tries to capture the formation of any arbitrary number of groups. Esteban and Ray (1991), Esteban and Ray (1994), Zhang and Kanbur (2001) and Duclos, Esteban and Ray (2004) belong to this first family of measures.<sup>3</sup> The second broad family conceives the existence of two groups only, with the median income as the divide. This family includes Foster and Wolfson (1992), Wolfson (1994) and Wang and Tsui (2000).<sup>4</sup>

We start by introducing the useful distinction between "polarization" and "bi-polarization", the latter restricting its scope to the eventual existence of

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<sup>1</sup>See Esteban and Ray (1991, 1994), Foster and Wolfson (1992), Wolfson (1994, 1997), Alesina and Spolaore (1997), Quah (1997), Wang and Tsui (2000), Esteban, Gradín and Ray (1998), Chakravarty and Majumder (2001), Zhang and Kanbur (2001) and Rodríguez and Salas (2002).

<sup>2</sup>See, for instance, D'Ambrosio and Wolff (2001), Collier and Hoeffler (2001), Fajnzylber, Lederman and Loayza (2000), Garcia-Montalvo and Reynal-Querol (2002), Gradín (2000), Knack and Keefer (2001), Milanovic (2000), Quah (1997) and Reynal-Querol (2002). See also Esteban and Ray (1999) for a formal analysis of the connections between polarization and the equilibrium level of conflict in a model of strategic interaction.

<sup>3</sup>García Montalvo and Reynal-Querol (2005) can also be associated with this family. Their measure does not depend on intergroup distance and hence it cannot be fairly compared with the other measures: Their case is equivalent to all group being equidistant to each other.

<sup>4</sup>Other measures in this family are Alesina and Spolaore (1997) and Rodríguez and Salas (2002).

two poles. In Section 3 we introduce a series of properties that capture features that a measure of polarization might be required to possess. These properties are the different axioms that have been put forward by Esteban and Ray (1991), Esteban and Ray (1994), and Duclos, Esteban and Ray (2004), on the one side, and by Foster and Wolfson (1992) and Wang and Tsui (2000) on the other. In both cases the axiomatization of the measure is but partial, as they depend on restricting the measure to belong to a pre-specified class of indices. We do not unpack these black-boxes, but launch instead three complementary axioms. In section 4 we focus on these polarization measures and check the subset of properties that each measure satisfies or fails to satisfy. This analysis provides a clear view of what these measures have in common and where they diverge. In an appendix we extend our analysis to three additional measures: Esteban, Gradin and Ray (1998), Zhang and Kanbur (2001) and Alesina and Spolaore (1997).

## 2 The Notion of Polarization: two views

As mentioned in the Introduction, the existing measures of polarization can be usefully classified into two broad families. One is designed to capture the formation of any arbitrary number of poles. We shall call them "measures of polarization" proper. The second family sees polarization as the process by which a distribution becomes *bi-polar*. We shall call them "measures of bi-polarization".

*Polarization measurement* starts with Esteban and Ray (1991) and Duclos, Esteban and Ray (2004) —DER thereafter— for continuous distributions and with Esteban and Ray (1994) —ER— for discrete distributions. An alternative measure of polarization has been proposed by Zhang and Kanbur (2001).

*Measures of bi-polarization* initiate with the work of Wolfson (1994) and (1997), based on Foster and Wolfson (1992), and by Wang and Tsui (2000) —WT thereafter. The measure of polarization put forward by Alesina and Spolaore (1997) should be considered as a member of this family.

There are three properties<sup>5</sup> that both families seem to consider to be indispensable to a measure of polarization:

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<sup>5</sup>See Esteban and Ray (1994) for an extensive motivation of these ideas, illustrated with numerous examples.

(i) polarization is a matter of groups so that when there is one group only there should be little polarization,

(ii) polarization raises when "within-group" inequality is reduced, and

(iii) polarization rises when "across-group" inequality increases.

Notice that the third claim runs directly against the ordering over distributions generated by second order stochastic dominance. It is plain that the notion of polarization is distinctly different from that of inequality in as much as we require inequality measures to be consistent with Lorenz curve orderings.

In order to have a precise idea of the distinguishing properties of each polarization measure, we shall examine whether they obey or not a series of Axioms intended to capture the essential features of polarization.

In the next section we present the collection of Axioms.

### 3 Axioms

Except when explicitly stated we shall work with distributions  $F$  in  $\mathfrak{R}_+$  with continuous density functions  $f$ .

We choose to present the different properties of polarization using the approach of DER where the statements of the Axioms are referred to distributions composed of "basic densities".

#### 3.1 Basic Densities

The properties —axioms— we shall deal with will largely be based on domains that are unions of one or more symmetric "basic densities." The densities will be scaled down and up as they may correspond to varying populations. The building block for these densities we will call *kernels*. These are symmetric, unimodal density functions  $f$  with compact support that we always situate on the interval  $[0, 2]$ . Their mean is one. By symmetry we mean that  $f(x) = f(2 - x)$  for all  $x \in [0, 1]$ , and by unimodality we mean that  $f$  is nondecreasing on  $[0, 1]$ . We normalize the overall population of a kernel to be unity.

A kernel  $f$  can be *population scaled* to any population  $p$  by multiplying  $f$  pointwise by  $p$  to generate a new density  $pf$ . Likewise, any kernel (or density) can undergo a *slide*. A *slide to the right by  $x$*  is just a new density  $g$

such that  $g(y) = f(y - x)$ . Similarly for a slide to the left. Further, a kernel  $f$  can be *income scaled* to any new mean  $\mu$  as follows:  $g(x) = (1/\mu)f(x/\mu)$  for all  $x$ . This will produce a new density  $g$  with support  $[0, 2\mu]$  and mean of  $\mu$ . Any scaling or slide (or combinations thereof) of a kernel we will call a *basic density*.

We shall also use the notion of a *squeeze*. Let  $f$  be any density with mean  $\mu$  and let  $\lambda$  lie in  $(0, 1]$ . A  $\lambda$ -*squeeze* of the density  $f$  is a transformation of this density as follows:

$$f^\lambda(x) \equiv \frac{1}{\lambda} f\left(\frac{x - [1 - \lambda]\mu}{\lambda}\right). \quad (1)$$

A  $\lambda$ -squeeze is a very special type of *second order stochastic dominance* transformation that contracts the support of a distribution towards its mean. Thus, a squeeze truly collapses a density inwards towards its mean. This has to be contrasted with arbitrary, unrestricted progressive Dalton transfers which can concentrate density around any point in the support of the distribution.

$\lambda$ -squeezes have the following properties:

- [P.1] For each  $\lambda \in (0, 1)$ ,  $f^\lambda$  is a density.
- [P.2] For each  $\lambda \in (0, 1)$ ,  $f^\lambda$  has the same mean as  $f$ .
- [P.3] If  $0 < \lambda < \lambda' < 1$ , then  $f^\lambda$  second-order stochastically dominates  $f^{\lambda'}$ .
- [P.4] As  $\lambda \downarrow 0$ ,  $f^\lambda$  converges weakly to the degenerate measure granting all weight to  $\mu$ .

Notice that a squeeze as defined could be applied to any density.

## 3.2 Axioms

Esteban and Ray (1991, 1994), and Duclos, Esteban and Ray (2004), and Foster and Wolfson (1992), Wolfson (1994) and Wang and Tsui (2000) have independently provide axioms for polarization and bi-polarization, respectively. As it turns out, although the spirit of the two sets of axioms is quite similar, they have significantly different implications. In order to stress the similarity in spirit of the different axioms, we shall present the WT counterparts —when they exist— with the same number as in DER, followed by a B as in 2B, for instance.

A second observation is that the complete axiomatizations by ER and DER, on one side, and of WT, on the other, both make use of an axiom that directly defines a class of polarization measures, different in each case. It is an open problem to disentangle how these axioms can be broken down into a collection of simpler statements. We shall instead posit three new axioms that permit a clear distinction between the different families of polarization measures.

We start by presenting the axioms in DER and in WT. Both papers contain a detailed motivation for these axioms and hence we simply refer the interested reader to these references for further discussion.

Axiom 1 tries to capture the first of the three properties.

**AXIOM 1.** If a distribution is composed of a *single* basic density, then a squeeze of that basic density cannot increase polarization.

Axiom 1 is quite uncontroversial. A squeeze, as defined here, corresponds to a compression of the entire basic density towards its mean and we must associate this to no higher polarization.

This Axiom, however, is less innocent than it looks at first sight. DER motivate their intuition for polarization as the outcome of the interplay between the individual sense of *group identification* combined with the feeling of *inter-personal alienation*. From this point of view, it is clear that Axiom 1 will generate some interesting restrictions on the measurement of polarization. This is so because, on the one hand, a squeeze creates a reduction in inter-individual alienation but, on the other, also serves to raise identification for a positive measure of agents — those located “centrally” in the distribution. The implied restriction is, then, that the latter’s positive impact on polarization must be counterbalanced by the former’s negative impact.

Our next axiom wishes to capture property (ii) above. It considers a situation composed of three disjoint densities, derived from identical kernels. The overall distribution is completely symmetric, with densities 1 and 3 having the same total population and with density 2 exactly midway between densities 1 and 3. We also assume that all supports are disjoint.

**AXIOM 2.** If a symmetric distribution is composed of three basic densities drawn from the same kernel, with mutually disjoint supports, then a symmetric squeeze of the *side* densities cannot reduce polarization.

In some sense, this is the defining axiom of polarization. This is precisely what we used to motivate the concept. This axiom argues that a particular “local” squeeze (as opposed to the “global” squeeze of the entire distribution in Axiom 1) must not bring polarization down. Here we explicitly depart from inequality measurement as it would predict that these local squeezes would reduce inequality.

Property (ii) has been specified by Foster and Wolfson (1992) and by Wang and Tsui (2000) somewhat differently. Bearing in mind that their aim is to measure bi-polarization, they take the median income  $m$ ,  $F(m) = \frac{1}{2}$ , as the reference point. Their specification of property (ii) is:

**AXIOM 2B [INCREASED BI-POLARITY]** Let distributions  $F$  and  $G$  have the same mean and the same median and let  $F$  second order stochastically dominate  $G$  separately on  $[0, m]$  and on  $[m, \infty)$ . Then  $F$  should be more polarized than  $G$ .

Notice that Axiom 2 is far less demanding than Axiom 2B. The latter requires that any two-sided increase of concentration in any distribution to be polarization increasing. In contrast Axiom 2 makes this requirement only for the particular case of local  $\lambda$ -squeezes and for the very special class of distributions described there. For the rest of distributions and for forms of local concentrations of density other than  $\lambda$ -squeezes Axiom 2 is silent.

Our third axiom materializes the idea behind property (iii) concerning an increase in “across-group” inequality. In the same line as in Axiom 2 we wish to specify this axiom in the least demanding way. To this effect we restrict to symmetric distributions composed of *four* non-overlapping basic densities, once again all generated by the same kernel.

**AXIOM 3.** Consider a symmetric distribution composed of four basic densities drawn from the same kernel, with mutually disjoint supports. An equal slide of the two inner densities outwards towards the outer densities makes polarization go up.

Here again Foster and Wolfson (1992) and Wang and Tsui (2000) present this idea differently. Their axiom asserts that if in a distribution everyone’s income is shifted farther away from the median income polarization goes up. For continuous distributions their axiom reads as:

AXIOM 3B [INCREASED SPREAD] Consider two distributions with the same mean and median such that  $|m - F^{-1}(p)| \leq |m - G^{-1}(p)|$ , for every  $p \in [0, 1]$ , then  $G$  is has more polarization than  $F$ .

Once again, Axiom 3 is much less demanding than Axiom 3B and hence should certainly be satisfied whenever Axiom 3B is. In Axiom 3 the claim that polarization should go up after all the groups have moved (weakly) farther away from the median (mean) is restricted to the special family of distributions described in the statement of that Axiom and the shift outwards is experienced only by the members of the two groups that are closer to the median.

Notice that Axiom 1 too is a special case of Axiom 3B. Inversely reading Axiom 3, it asserts that if a symmetric, unimodal "basic density" is  $\lambda$ -expanded —the opposite of squeezed— polarization cannot come down. It is straightforward that a  $\lambda$ -expansion is a specific form of "increased spread" as defined in Axiom 3B.

From the previous discussion it follows that if a polarization measure satisfies Axiom 3B it must also satisfy Axioms 1 and 3.

Our fourth axiom is a simple population-invariance principle. It states that if one distribution is more polarized than another, it must continue to be so when the populations in both situations are scaled up or down by the same amount, leaving all (relative) distributions unchanged.

AXIOM 4. Let  $F$  and  $G$  be two distributions with possibly different, un-normalized populations such that  $P(F) \geq P(G)$ . Then, for all  $\kappa > 0$ ,  $P(\kappa F) \geq P(\kappa G)$ , where  $\kappa F$  and  $\kappa G$  represent (identical) population scalings of  $F$  and  $G$  respectively.

Wang and Tsui (2000) propose a scale invariance axiom that serves a similar purpose as Axiom 4. However, while Axiom 4 posits population invariance, Wang and Tsui require income invariance with respect to the median. That is:

AXIOM 4B [SCALE INVARIANCE] Let the distributions  $F$  and  $G$  have the same median income  $m$ . Then if  $G(x)$  is more polarized than  $F(x)$ , so is  $G(\frac{x}{m})$  relative to  $F(\frac{x}{m})$ .

The fifth Axiom posits that polarization indices have to belong to a particular class, that is

AXIOM 5 Let  $F$  be a distribution in  $\mathfrak{R}_+$  with a continuous density  $f$ . Then

$$P(F) = \int \int T(f(x), |x - y|) f(x)f(y)dx dy, \quad (2)$$

where  $T$  is some function increasing in its second argument and with  $T(0, a) = T(i, 0) = 0$ .

In a similar vein as in Esteban and Ray (1994), DER motivate this assumption on the basis that "aggregate" polarization has to be conceived as the sum of all inter-personal *effective antagonisms*. Interpersonal antagonisms  $T$  are assumed to result from the own sense of *identity*  $i$  —which in turn depends on the group size  $i = f(x)$ — and from inter-personal *alienation*  $a$  —which depends on income distance  $a = |x - y|$ . Hence,

$$T(i, a) = T(f(x), |x - y|). \quad (3)$$

Continuing with the parallelism between DER and Wang and Tsui (2000), they also restrict polarization indices to belong to a specific additive class:

AXIOM 5B Let  $F$  be a distribution in  $\mathfrak{R}_+$  with a continuous density  $f$ . Then, the polarization measure should be of the form

$$P(F) = \frac{1}{m} \int a(F(x))xf(x)dx. \quad (4)$$

Whereas Axioms 1 to 4 —including their B counterparts— consist of simple assertions concerning changes in the distribution, Axioms 5 and 5B are a bit of a black-box. We shall not attempt at decomposing them into a collection of simpler statements. We shall instead posit three additional properties that will permit to identify the differential behavior of the various indices.

### 3.3 Additional Properties

We start with what has become the most standard motivating example of the differences between polarization and inequality. This is that an increase in concentration around the mean of a distribution will reduce both inequality and polarization, while if this concentration takes place around several poles polarization increases —and inequality still decreases. Our Axiom 1 captures

the first type of concentration around the mean income. It remains to be examined the intuition that when this concentration takes place around two or more poles polarization should increase. As we shall now see, in this case we need to be more careful when stating the conditions under which it is reasonable to presume that polarization should go up.

Suppose that we partition the (bounded) support of a distribution in a number of intervals and that we concentrate the population within each interval. Here too we need to make sure that within each interval population is concentrated around the center rather than around sub-poles. To this effect, we will concentrate the population within each interval by means of a  $\lambda$ -squeeze. Now observe that our judgement on the effects of a  $\lambda$ -squeeze should be conditioned on the shape of the distribution. To see this, consider a U-shaped global distribution and perform a  $\lambda$ -squeeze on each side of the mean. Now, the spikes at the extreme of the support will come closer together and mass will be concentrated around the two (conditional) means. In that case one would rather think that polarization has even come down. Thus, in order to find an uncontroversial case in which forming various bunches of population in a distribution increases polarization we need to restrict the type of the initial distribution.

The following case seems to be quite uncontroversial. Take a uniform distribution over a bounded support  $[a, b]$ . Consider now a partition of the support into  $n$  intervals of equal length  $\frac{b-a}{n}$ . Thus, the entire distribution can be seen as  $n$  identical uniform densities with non-overlapping support, each one with a mass  $\frac{1}{n}$ . Suppose now that every such uniform density is subject to a  $\lambda$ -squeeze. Intuition clearly says that polarization should increase since we have moved from a uniform distribution —without any pole— to a distribution displaying  $n$  poles.

Notice that the scenario described in Axiom 2 is similar to this one. The three basic densities assumed there can be taken to be uniform. However, for the case of three intervals, the assertion in Axiom 2 is somewhat stronger for there the outer distributions only are subject to a squeeze, while here we are squeezing all the three uniform densities. Observe further that in this case, the squeezing of the two outer distribution only does satisfy the conditions of Axiom 2B: second order stochastic dominance on each side of the median. However, this condition is not met when the three densities are squeezed because this entails a progressive transfer of income across the median.

With this proviso, we move on to formally introduce our next Axiom:

AXIOM 6 Consider a uniform distribution with support  $[a, b]$ . Let us partition this support into  $n$  intervals of length  $\frac{b-a}{n}$ . Then, a  $\lambda$ -squeeze of the  $n$  uniform densities should (weakly) increase polarization.

To motivate our next Axiom, consider the argument in Esteban and Ray [1994] according to which any reasonable measure of polarization should display fundamental non-monotonicities. This is in contrast with the property of the Lorenz-curve ordering that if distribution  $F$  dominates  $G$ ,  $F$  can be obtained from  $G$  by means of a sequence of Daltonian progressive income transfers.

In order to specify the point, let us consider a situation similar to the one contemplated in Axiom 3 with a symmetric distribution consisting of four non-overlapping densities. Yet, instead of shifting outwards the inner densities as in there, we now symmetrically transfer population mass bit by bit from the inner densities towards the outer densities. We thus will start with most of the total population located in the inner densities and will end with most of the population located in the outer densities. Clearly, the final distribution is more polarized than the initial one.

The important question, though, is how polarization should behave in the intermediary steps. As we transfer population from the inner densities towards the outer densities we increase the alienation felt by every individual with respect to the rest. However, at the same time the distribution starts showing four, rather than two, groups and we expect this to reduce the sense of group identity. Thus, if the outer groups are not too far from the inner groups so as to limit the effects of increased alienation, this process of transferring population towards the outer densities should have to have a non-monotonic effect on polarization, with an initial decrease followed by a final increase. Think, for instance, of the situation in which the process is half the way so that the distribution consists of four densities of equal mass. It seems acceptable—even desirable—that polarization is measured here (or at some point in the transfer process) to be lower than in the initial state. But, of course, if the population of the inner densities is shifted sufficiently far away, the increase in alienation will eventually overcome any loss in identification, thus making polarization to go up all the way. We formalize these ideas in the following Axiom.

AXIOM 7 Consider a symmetric distribution like the one described in Axiom 3 with four non-overlapping basic densities deriving from the same

kernel and with each of the outer densities having population  $\delta$  and the inner densities  $\frac{1}{2} - \delta$ . Then, if the distance between the inner and outer densities is sufficiently small, polarization should not vary monotonically with  $\delta$ .

Observe that Axiom 7 is not compatible with Axiom 3B. In the scenario contemplated by Axiom 7 the sequence of distributions generated by transferring population from the inner to the outer densities satisfies the condition required by Axiom 3B because the distance with respect to the median is pointwise uniformly larger. In this situation, Axiom 3B requires the sequence of distributions to display monotonically increasing polarization while Axiom 7 requires that the recorded polarization be non-monotonic. There is no measure that could simultaneously satisfy both demands.

Axiom 7 stresses the fact that polarization aims at capturing the existence of identifiable groups and their size and distance and not poverty or inequality. Consequently, it seems reasonable to require that our judgement on how polarized is a distribution with two groups of size  $p$  and  $1 - p$  and at a distance  $d$  of each other, for instance, should not depend on which of the two is rich or poor. Permuting positions should leave polarization unchanged (but not the recorded inequality!). More generally, we consider that the flipping of a distribution around the mid-point of its support should leave polarization unchanged. Suppose we are examining the polarization of a distribution of political locations on an interval. Symmetrical situations should lead to the same degree of polarization, irrespective of where left and right are located on the line. The same can be said, for instance, of the degree of polarization of the ethnic composition of a given society (or of the religious beliefs).

Thus, we posit the following property.

**AXIOM 8** Take any arbitrary density  $f(x)$  with support  $[a, b]$ . Consider now the density  $g(x)$  with the property that  $g(x) = f(2M - x)$ , where  $M = \frac{a+b}{2}$ . Then, polarization under  $f$  and under  $g$  should be the same.

## 4 Comparing Polarization Measures

### 4.1 Measures of Polarization

Based on the view that polarization is the outcome of inter-personal alienation fueled by the sense of identification, the polarization measure obtained

by Esteban and Ray (1991) and Duclos, Esteban and Ray (2004) is

$$P^{DER} = \int \int f(x)^{1+\alpha} f(y) |y - x| dy dx, \quad (5)$$

where  $\alpha \in [0.25, 1]$  and the distribution has been normalized to a mean income of unity.

This measure is to be compared with the equivalent one in Esteban and Ray (1994) designed to measure polarization for discrete distributions. A discrete distribution is characterized by a  $n$ -dimensional vector of incomes  $x_i$  and population shares  $p_i$ ,  $i = 1, \dots, n$ . Their measure then reads

$$P^{ER} = \sum \sum p_i^{1+\alpha} p_j |x_i - x_j|, \quad (6)$$

with  $\alpha \in (0, 1.6]$ .

In this case, the "within-group" component referred to above is entirely absent. Polarization depends on the number and size of population groups and the inter-group distances.

**Proposition 1** *The polarization measure  $P^{DER}$*

(i) *satisfies Axioms 1 to 5 if and only if it can be written as in (5);*

(ii) *satisfies Axioms 4B, and 6 to 8; and*

(iii) *fails to satisfy Axioms 2B, 3B and 5B.*

*Proof.- Part (i) is the characterization Theorem in DER. The interested reader can verify the details of the proof there.*

*With respect to Part (ii), we start by showing that Axiom 4B is satisfied by (5). The reason is simple. The characterization Theorem in DER demonstrates that the measure that corresponds to the five axioms above is proportional to (5). In the discussion following, DER show that choosing the free scaling parameter to be  $\mu^{1-\alpha}$  is equivalent to normalizing the distribution to a mean income of unity. Following the very same steps one can obtain that choosing this parameter to be  $m^{1-\alpha}$  is equivalent to normalize incomes to the median income,  $m$ . It is straightforward that (5) normalized to the median income does satisfy Axiom 4B.*

*The proof that (5) satisfies Axioms 6 and 7 is a bit intricate. We start with the intuition as to why  $P^{DER}$  satisfies these axioms..*

*The intuition of why (5) satisfies Axiom 6 is as follows. We know that this measure satisfies Axiom 2. Hence, in any symmetric distribution with*

three basic densities the squeezing of the two outer densities should increase polarization. Notice that uniform densities are a particular class of basic densities. Moreover, we are free to choose the population weight of the density in the middle as well as the width of its support. Hence, we can choose both population and support of the middle density to be vanishingly small. Then we are left with a scenario that can be made arbitrarily close to two uniform distributions with nearly adjacent support. Because of Axiom 2, the squeezing of these distributions should bring polarization up. The final step is to realize that the fact that polarization goes up is not conditional on the number of adjacent uniform densities that are being squeezed.

In order to prove that  $P^{DER}$  satisfies Axiom 6, we consider a uniform distribution over  $[a, b]$ . We shall partition this support into  $n$  identical intervals of support  $[a_i, b_i]$  with length  $\frac{b-a}{n}$  and population of  $\frac{1}{n}$ . Within each interval we have an (unnormalized) uniform distribution with density  $\frac{1}{b-a}$ . Notice that

$$a_i = (i-1)\frac{b-a}{n} \text{ and } \mu_i = (i-1)\frac{b-a}{n} + \frac{b-a}{2n}.$$

We shall subject all these  $n$  identical densities to a  $\lambda$ -squeeze.

We can conceive this scenario as a distribution made of  $n$  basic densities  $f$  obtained from the sliding of the same root,  $f^*$ , this being a uniform distribution on  $[0, 2]$  with density  $\frac{1}{2}$ . Thus we can follow the same procedure as in Lemma 2 (and followings) in DER (pp. 1761 and ff) and decompose total polarization as the sum of all the "within-density" polarization,  $W_i$ , and all the "across-density" effective antagonisms,  $A_{ij}$ .

Since all the basic densities are identical replicas of the same root, we can use Lemma 6 in DER. Notice that  $p = \frac{1}{n}$ ,  $s_i \equiv \mu_i - a_i = \frac{b-a}{2n}$  to obtain

$$\begin{aligned} \sum_{i=1}^n W_i &= n4n^{-2-\alpha} \left(\frac{b-a}{2n}\right)^{1-\alpha} \lambda^{1-\alpha} \int_0^1 f^*(x)^{1+\alpha} \left\{ \int_0^1 f^*(y)(1-y)dy + \int_x^1 f^*(y)(y-x)dy \right\} dx \\ &= 2^{1+\alpha}(b-a)^{1-\alpha} n^{-2} \lambda^{1-\alpha} \int_0^1 2^{-1-\alpha} \left\{ \int_0^1 2^{-1}(1-y)dy + \int_x^1 2^{-1}(y-x)dy \right\} dx = \\ &= 2^{-1}(b-a)^{1-\alpha} n^{-2} \lambda^{1-\alpha} \int_0^1 \left\{ \int_0^1 (1-y)dy + \int_x^1 (y-x)dy \right\} dx = \\ &= \frac{1}{3} (b-a)^{1-\alpha} n^{-2} \lambda^{1-\alpha}. \end{aligned}$$

Let us now compute  $A_{ij}$ . Using Lemma 7 in DER we have

$$A_{ij} = 2|\mu_i - \mu_j|n^{-2-\alpha} \left(\frac{b-a}{2n}\right)^{-\alpha} \lambda^{-\alpha} \int_0^1 f^*(x)^{1+\alpha} dx = \quad (9)$$

$$= |i-j|n^{-3} (b-a)^{1-\alpha} \lambda^{-\alpha}. \quad (10)$$

Therefore, adding all the cross antagonisms we obtain

$$A = n^{-3} (b-a)^{1-\alpha} \lambda^{-\alpha} \sum_i \sum_j |i-j|. \quad (11)$$

Thus, for total polarization we finally obtain

$$P^{DER}(\lambda) = W + A = \frac{1}{3} (b-a)^{1-\alpha} n^{-2} \lambda^{1-\alpha} + n^{-3} (b-a)^{1-\alpha} \lambda^{-\alpha} \sum_i \sum_j |i-j|. \quad (12)$$

Differentiating (5) with respect to  $\lambda$  we have

$$\frac{\partial P^{DER}}{\partial \lambda} = (b-a)^{1-\alpha} n^{-2} \lambda^{-1-\alpha} \left( \frac{1-\alpha}{3} \lambda - \alpha \frac{\zeta(n)}{n} \right), \quad (13)$$

$$\text{where } \zeta(n) \equiv \sum_i \sum_j |i-j|.$$

The sign of this derivative depends on the term in braces only, which we denote by  $B$ . Clearly

$$B \leq \left\{ \frac{(1-\alpha)}{3} - \alpha \frac{\zeta(n)}{n} \right\}$$

Observe now that  $\frac{\zeta(n)}{n}$  is increasing in  $n$  and that  $\frac{\zeta(2)}{2} = 1^6$ . Hence,  $B \leq (1-4\alpha) \leq 0$  for  $\alpha \geq 0.25$ .

It follows that for  $\alpha \geq 0.25$  a  $\lambda$ -squeeze —that is, a decrease in  $\lambda$ — will increase polarization for every partition possible into  $n$  intervals of the support of a uniform distribution. Therefore (5) does satisfy Axiom 6.

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<sup>6</sup>Observe first that  $\zeta(n+1) = \zeta(n) + n(n+1)$ . Hence,  $\frac{\zeta(n+1)}{n+1} - \frac{\zeta(n)}{n} = n - \frac{\zeta(n)}{n(n+1)}$ .

We know that  $\zeta(n) \leq n \cdot n^2 = n^3$ . Thus,

$$\frac{\zeta(n+1)}{n+1} - \frac{\zeta(n)}{n} \geq \frac{n}{(n+1)} > 0.$$

Let us now turn to Axiom 7. The intuition that (5) satisfies Axiom 7 can be conceived in two steps: (i) after having shifted all the population from the inner to the outer densities polarization should be higher than at the beginning of the process when no population had been shifted yet and (ii) when all the population is at the inner densities the beginning of the transfer of population to the outer densities makes polarization come down. For the first step, consider the symmetric distribution made of four densities with non-overlapping support with the inner densities endowed with a population  $q$  and the outer densities with population  $p$ , at a distance  $d$  from each other. The first step asserts that polarization is higher at the end of the transferring process when the outer densities have population  $p + q$  than at the beginning when all the population is shared by the inner densities. Notice that this is just like shifting the inner densities outwards by a distance  $d$ . Suppose now that together with the four densities there were two more densities farther out with a population  $r$ . Then, by Axiom 3, the shifting of the inner densities — with mass  $p + q$  — towards the position of the outer densities should increase polarization even for vanishingly small  $r$ . The second step, requires showing that at the initial position with all the population located at the inner densities polarization comes down as we start shifting population towards the outer locations.

We start with the four density configuration in which the mass of each of the inner densities is  $\frac{1}{2} - \delta$  and that of each of the outer densities is  $\delta$ ,  $0 < \delta < 1$ . We wish to examine the behavior of  $P^{DER}$  as  $\delta$  increases. In order to compute  $P^{DER}$  we again decompose total polarization into the sum of within-density polarization and across-density antagonisms.

We start with the within-density polarization. Because of the symmetry of the distribution we just need to focus on the inner and the outer densities,  $P_i(\delta)$  and  $P_o(\delta)$  respectively. Once again, using Lemma 6 in DER together with the fact that  $\mu_i - a_i = s$ ,  $i = 1, 2, 3, 4$ , and that  $d_{oi} = d$  and  $d_{ii} = D$ , we obtain

$$P^{DER}(\delta) = 2W_i(\delta) + 2W_o(\delta) + \sum_i \sum_j A_{ij} = \tag{14}$$

$$= K \left\{ \begin{array}{l} \left[ \left( \frac{1}{2} - \delta \right)^{2+\alpha} + \delta^{2+\alpha} \right] 2s + \\ \left[ \left( \frac{1}{2} - \delta \right)^{1+\alpha} \frac{4\delta d + D}{2} + \delta^{1+\alpha} \frac{2d + D}{2} \right] \psi(f, \alpha) \end{array} \right\}, \tag{15}$$

with

$$K = 4s^{-\alpha} \int_0^1 f^*(x)^{1+\alpha} \left\{ \int_0^1 f^*(y)(1-y)dy + \int_x^1 f^*(y)(y-x)dy \right\} dx \quad (16)$$

and

$$\psi(f, \alpha) = \frac{\int_0^1 f^*(x)^{1+\alpha} dx}{\int_0^1 f^*(x)^{1+\alpha} \left\{ \int_0^1 f^*(y)(1-y)dy + \int_x^1 f^*(y)(y-x)dy \right\} dx} \quad (17)$$

as in (39) in Lemma 8 of DER.

It is straightforward to verify that  $P^{DER}(\frac{1}{2}) > P^{DER}(0)$ .

In order to prove the non-monotonicity of  $P^{DER}$  with respect to  $\delta$  it suffices to show that it is strictly decreasing at  $\delta = 0$ .

Differentiating with respect to  $\delta$  we have

$$\frac{\partial P^{DER}}{\partial \delta} = K \left\{ \begin{aligned} & \left[ -(2+\alpha) \left( \frac{1}{2} - \delta \right)^{1+\alpha} + (2+\alpha) \delta^{1+\alpha} \right] 2s + \\ & + \left[ \left( \frac{1}{2} - \delta \right)^\alpha \left( \frac{2d-4\delta d}{2} - (1+\alpha) \frac{4\delta d+D}{2} \right) + (1+\alpha) \delta^\alpha \frac{2d+D}{2} \right] \psi(f, \alpha) \end{aligned} \right\}. \quad (18)$$

When we evaluate it at  $\delta = 0$  we have

$$\frac{\partial P^{DER}}{\partial \delta} \Big|_{\delta=0} = K \left( \frac{1}{2} \right)^\alpha \left\{ -(2+\alpha)2s + \left[ d - (1+\alpha) \frac{D}{2} \right] \psi(f, \alpha) \right\}.$$

Since  $s$  can be arbitrarily small without this imposing any restriction on  $\psi(f, \alpha)$ , we have that  $d < \frac{1+\alpha}{2}D$  (or  $d < \frac{5}{8}D$  for any admissible  $\alpha$ ) is sufficient for polarization to decrease at the early stages of the transferring of population from inner densities towards outer densities.<sup>7</sup> It follows that (5) satisfies Axiom 7.

Let us finally show that (5) satisfies Axiom 8. This is quite straightforward. In this case, we have a new distribution with density  $g(x)$  with the property that  $g(x) = f(2M - x)$ , where  $M = \frac{a+b}{2}$ . Using this equality in (5) and

<sup>7</sup>It can be easily computed that  $\frac{\partial P}{\partial \delta} \Big|_{\delta=\frac{1}{4}} < 0$ .

performing the change of variables  $x' = 2M - x$  and  $y' = 2M - y$  it is immediate that  $P(G) = P(F)$ . This completes the proof of part (ii).

Part (iii) states that (5) fails to satisfy Axioms 2B, 3B and 5B. Let us start by the latter. Clearly, Axiom 5B cannot be satisfied by (5) because it is Axiom 5 the one that together with the other four axioms uniquely determines  $P^{DER}$  polarization measure.

Let us now consider Axiom 3B. We have already mentioned that Axioms 1 and 3 are weaker than Axiom 3B. However, it is interesting to note that the gap between Axiom 3B and the union of Axioms 1 and 3 is critical for the properties of the derived measures. We have proven that Axiom 7 is compatible with Axioms 1 and 3 because they are all satisfied by (5). Now simply note that Axiom 7 cannot be compatible with Axiom 3B, for the latter requires monotonicity of the polarization measure with the increase of distances with respect to the median/mean income. Hence,  $P^{DER}$  fails to satisfy Axiom 3B.

Let us finally show that (5) does not satisfy Axiom 2B (while satisfying Axiom 2!). To this effect, let us consider a symmetric distribution consisting of six basic densities obtained from the same kernel. Now the population mass of each density will be  $(\frac{1}{4} - \delta, 2\delta, \frac{1}{4} - \delta, 2\delta, \frac{1}{4} - \delta)$ , with  $\delta \in [0, \frac{1}{4}]$  and  $\mu_i - a_i = s$ ,  $i = 1, \dots, 6$ . When  $\delta = 0$  we have a distribution with the population concentrated in four basic densities and when  $\delta = \frac{1}{4}$  we have the population concentrated in two densities each located half way between the two densities on each side of the median/mean. Observe now that increases in  $\delta$  imply progressive Dalton transfers within the population below and within the population above the median/mean income. On each side of the median we are decreasing the population size of the richest and poorest densities to the benefit of the group half way in between. While for most reasonable polarization measures polarization should be definitely higher at the end of the process than at the beginning, polarization needs not monotonically increase with  $\delta$ . Consider simply the case with  $\delta = \frac{1}{12}$  where all densities have the same population weight of  $\frac{1}{6}$ . The same intuition behind Axiom 7 applies here: the creation of extra groups should eventually be detrimental to polarization.

As before, we shall compute  $P(\delta)^{DER}$  as the sum of the inner polarization within each density and the sum of the effective antagonisms across densities. We shall continue to use Lemmas 6 and 7 in DER.

We start with the inner polarization  $W$ . There are four identical densities with population mass  $\frac{1}{4} - \delta$  and two with mass  $2\delta$ . Hence, using Lemma 6 we

have

$$W = 8s^{1-\alpha}K \left\{ (2\delta)^{2+\alpha} + 2 \left( \frac{1}{4} - \delta \right)^{2+\alpha} \right\}.$$

Since the distribution is symmetric, the across group antagonism  $A$  will be twice the sum of antagonisms of the three groups with mean income below the median/mean,  $A_1, A_2, A_3$ . Using Lemma 7 in DER we obtain

$$\begin{aligned} A_1 &= 2 \left( \frac{1}{4} - \delta \right)^{1+\alpha} s^{-\alpha} K \psi(f, \alpha) \left( 2d + \frac{D}{2} \right), \\ A_2 &= 2(2\delta)^{1+\alpha} s^{-\alpha} K \psi(f, \alpha) \left( \frac{3}{2}d - 2d\delta + \frac{D}{2} \right), \\ A_3 &= 2 \left( \frac{1}{4} - \delta \right)^{1+\alpha} s^{-\alpha} K \psi(f, \alpha) \left( d + \frac{D}{2} \right). \end{aligned}$$

Adding all the components and rearranging, we have for the aggregate polarization

$$P^{DER}(\delta) = 4Ks^{-\alpha} \left\{ \begin{aligned} &2 \left( \frac{1}{4} - \delta \right)^{1+\alpha} \left( 2 \left( \frac{1}{4} - \delta \right) s + \psi(f, \alpha) \frac{1}{2} (3d + D) \right) + \\ &+ (2\delta)^{1+\alpha} \left( 2\delta s + \psi(f, \alpha) \frac{1}{2} (3d + D - 4d\delta) \right) \end{aligned} \right\}.$$

Differentiating  $P(\delta)^{DER}$  with respect to  $\delta$  and evaluating it at  $\delta = 0$  we finally obtain

$$\frac{\partial P^{DER}(\delta)}{\partial \delta} \Big|_{\delta=0} = 4Km^{-\alpha} \left\{ -\frac{1+\alpha}{2} \left( \frac{s}{2} + \psi(f, \alpha) \frac{1}{2} (3d + D) \right) - s4^{-\alpha} \right\} < 0.$$

Therefore, when  $\delta = 0$ , i.e. at the beginning of the process, the shifting of mass to create an intermediate density does decrease polarization as measured by  $P^{DER}$ , violating the claim in Axiom 2B.

## 4.2 Measures of Bi-Polarization

The two measures of bi-polarization we shall review share a common feature: both measures are different ways of computing the distance from any particular distribution to the symmetric bi-modal located at the extremes of the support. Conversely, they can be seen as different ways of measuring the distance with respect to the distribution with all the population concentrated at the median income.

Measures of bi-polarization initiate with the work of Wolfson (1994) and (1997), based on Foster and Wolfson (1992). Alternative measures have been proposed by Wang and Tsui (2000) and by Alesina and Spolaore (1997) [see the Appendix]. We shall present the polarization indices for the case of continuous distributions  $F$  on  $\mathfrak{R}_+$ , with  $f$  as the corresponding density.

The measure of bi-polarization in Wolfson (1994) can be written as

$$P^W = \frac{\mu}{m} \left[ \frac{1}{2} - L - \frac{G}{2} \right], \quad (19)$$

where  $m$  stands for the median income,  $G$  for the Gini index and  $L = L(\frac{1}{2})$  for the value of the ordinate of the Lorenz curve at the median income.

We wish to contrast now this measure with the set of axioms introduced in the previous section.

**Proposition 2** *Wolfson's polarization measure  $P^W$*

(i) *satisfies Axioms 2B, 3B and 4B —and a fortiori Axioms 1, 2, 3— and Axiom 4; and*

(ii) *it does not satisfy Axioms 6, 7 and 8.*

*Proof.- Wang and Tsui (2000) have proven that  $P^W$  satisfies Axioms 2B, 3B and 4B. It follows that it also satisfies Axioms 1, 2, and 3. Since  $P^W$  is population normalized, it also satisfies Axiom 4.*

*Let us now turn to part (ii) in the statement of our Proposition. In the case of Axiom 6, we start with a uniform distribution on  $[a, b]$ . This distribution has a density  $f(x) = \frac{1}{b-a}$  and a mean/median  $\mu = m = \frac{a+b}{2}$ . We wish to see the effect on  $P^W$  of a  $\lambda$ -squeeze on a uniform distribution partitioned into  $n$  equally sized intervals. Let us first suppose that  $n$  is even. In that case, the squeeze would leave the value of  $L$  unchanged —because no income would have been transferred across the mean/median of the distribution— and  $G$  would be reduced. It follows that  $P^W$  would increase. Let us now examine the case of an odd  $n$ . Now, while  $G$  continues to come down,  $L$  would go up because the squeeze of the interval containing the mean/median income in its interior would transfer income from the higher to the lower incomes in this interval. The two changes would shift  $P^W$  in opposite directions. In order to sign the net effect we need to compute both changes explicitly. To this we turn now.*

*We shall denote by  $L(n, \lambda)$  the share of total income below the mean/median income when the  $n$  intervals of a uniform distribution are subject to a  $\lambda$ -*

squeeze. It is immediate that we should focus only on the interval whose conditional mean/median income coincides with the overall mean/median income. Since the support of the overall uniform distribution is  $[a, b]$ , the length of any interval will be  $\frac{b-a}{n}$  and its mass  $\frac{1}{n}$ . We are interested in the share of total income of individuals in the interval  $[\mu - \frac{b-a}{2n}, \mu]$  as the distribution is  $\lambda$ -squeezed.

To this end we first observe that  $L(n, \lambda) = L_- + L_m(\lambda)$ , where the first term stands for the share contributed by the intervals below the median interval and  $L_m$  for the contribution of the median interval up to its median/mean income. When computing  $L_m(\lambda)$  we have to keep in mind that the relevant support is  $[\mu - \frac{b-a}{2n}, \mu]$ , that is  $[\frac{a+b}{2} - \frac{b-a}{2n}, \frac{a+b}{2}]$ . Then we have

$$L = L_- + L_m(\lambda) = \quad (20)$$

$$= \sum_{i=1}^{m-1} \frac{\mu_i}{n} + \int_{\frac{a+b}{2} - \frac{b-a}{2n}}^{\frac{a+b}{2}} \frac{\lambda 2x + (1-\lambda)(a+b)}{a+b} \frac{1}{b-a} dx = \quad (21)$$

$$= \sum_{i=1}^{m-1} \frac{\mu_i}{n} + \frac{1}{2n} - \frac{\lambda}{4n^2} \frac{b-a}{a+b}. \quad (22)$$

Differentiating with respect to  $\lambda$  we have

$$\frac{\partial L}{\partial \lambda} = -\frac{1}{4n^2} \frac{b-a}{a+b}. \quad (23)$$

In order to compute the effect of the  $\lambda$ -squeeze on  $G$  we can use the results obtained for  $P^{DER}$ . Indeed,  $G$  is equal to  $\frac{1}{2}P^{DER}$  when evaluated at  $\alpha = 0$ . [Notice that for  $P^{DER}$  to satisfy the corresponding axioms it must be that  $\alpha \geq 0.25$ ]. Using (13) evaluated at  $\alpha = 0$ , we have that

$$\frac{\partial G}{\partial \lambda} = \frac{1}{3n^2} \frac{b-a}{a+b}. \quad (24)$$

Differentiating (19) with respect to  $\lambda$  and using (23) and (24) we obtain

$$\frac{\partial P^W}{\partial \lambda} = -\frac{\mu}{m} \left[ \frac{1}{6n^2} \frac{b-a}{a+b} - \frac{1}{4n^2} \frac{b-a}{a+b} \right] = \frac{1}{2n^2} \frac{b-a}{a+b} > 0. \quad (25)$$

It follows that, when  $n$  is odd, a  $\lambda$ -squeeze will decrease  $P^W$  violating Axiom 6.

As we have already pointed out the fact that  $P^W$  satisfies Axiom 3B implies that it violates Axiom 7.

We finally show that  $P^W$  does not satisfy Axiom 8 either. Consider the following example. We have a degenerate distribution with  $\frac{2}{3}$  of the population with  $y_1 = 0$  and  $\frac{1}{3}$  with  $y_2 = 1$ . It can be readily verified that  $P^W$  is larger than for the distribution resulting from the flip of the initial one. This completes the proof.

Summarizing we have verified that Wolfson's measure satisfies the four Axioms characterizing  $P^{DER}$ , but fails to perform accordingly with Axioms 6, 7 and 8.

Let us stress the unhappy feature of Wolfson's measure that the effect on polarization of the making of the groups more cohesive depends on whether the number of such groups is odd or even. This is not surprising after one realizes that Wolfson's measure can be better understood as a measure of bi-polarization. Specifically, it can be seen as a measure of distance with respect to the most bi-polarized distribution: the one with the population concentrated on two spikes located at the extremes of the support of the distribution. Therefore, the bunching of probability around an odd number of groups takes that distribution away from the extreme bi-modal.

The second measure of bi-polarization we wish to examine is the one axiomatized by Wang and Tsui (2000). Their measure is

$$P^{WT} = P(F) = k \int \left| \frac{x - m}{m} \right|^r f(x) dx, \quad (26)$$

with  $r \in (0, 1)$ .

Accordingly with this measure, polarization is captured by the average of a concave transformation of the distance with respect to the median income. It essentially is a measure of distance to the polarization minimizing distribution with all the population concentrated at the median income.

**Proposition 3** Consider the polarization measure  $P^{WT}$ ,

- (i) A polarization measure satisfies Axioms 2B to 5B —and hence Axioms 1, 2 and 3— if and only if it can be written as (26);
- (ii) The polarization measure  $P^{WT}$  (26) does satisfy Axiom 4 but violates Axioms 6 to 8.

*Proof.- Part (i) of the Proposition has been proven by Wang and Tsui (2000, Proposition 6, p. 360).*

*Since  $P^{WT}$  is normalized with respect to total population, it does satisfy Axiom 4.*

*By the same reasons as before  $P^{WT}$  violates Axiom 7.*

*It remains to be shown that  $P^{WT}$  violates Axioms 6 and 8.*

*We start with an example of violation of Axiom 8. Suppose a degenerate distribution with  $\frac{1}{3}$  of the population in each of the three incomes (0, 1, 4). It is immediate to compute that  $P^{WT} = \frac{4}{3}$ . Flipping the previous distribution would give incomes (0, 3, 4) with a polarization of  $P^{WT} = \frac{4}{9} < P^{WT}$ .*

*We finally turn to the violation of Axiom 6. Consider the case of the partition of a uniform distribution into three equal intervals. This situation accomodates itself to the scenario described in Axiom 2. The effect of a squeeze on  $P^{WT}$  can be seen as composed by the effect of the squeezing of the two side densities and the effect of squeezing the inner density. By Axiom 2 the former will raise  $P^{WT}$  while the latter —by Axiom 1— will reduce it. We need to show an example where this second effect dominates over the first and makes overall  $P^{WT}$  fall. Consider the following example: a uniform distribution with support  $[0, 12]$  is partitioned into three uniform densities with supports  $[0, 4]$ ,  $[4, 8]$ ,  $[8, 12]$ . We now squeeze them and obtain new uniform densities with supports  $[1, 3]$ ,  $[5, 7]$ ,  $[9, 11]$ . It is a matter of computation to obtain that  $P^{WT}$  will be smaller after the squeeze. To conclude this point, let us simply note that if the partition of the original distribution had been into an even number of intervals, the squeeze of each uniform density would have produced an increase in  $P^{WT}$ . This concludes the proof.*

The measure proposed by Wang and Tsui behaves quite similarly to Wolfson's measure in that both violate the same set of axioms.

## 5 Summary of results

We have reviewed a number of alternative measures of polarization, grouped into two families: polarization and bi-polarization measures. Taking as a base the set of axioms that characterize the  $P^{DER}$  measure of polarization, we have demonstrated that the four basic axioms in DER are satisfied by all the measures of either family. It follows that these axioms capture what

is in common in all the polarization measures. However, we have presented three new axioms that, while still satisfied by  $P^{DER}$  are violated by the two bi-polarization measures that we have considered here.

Where the two families depart? We cannot provide a complete answer to this question. However, this paper goes one step towards a better understanding of the differences between the two families of polarization measures. In view of our results, there are two lines of departure. One line is that the bi-polarization measures satisfy axioms that are far stronger than the ones proposed in DER, albeit responding to the same intuition. The second, and most important, line is in the assumed "class of measures" as described in Axioms 5 and 5B.

Unfortunately, both 5 and 5B Axioms are a blackbox. ER (and DER) provide a "behavioral" motivation for their family of measures, based in the interaction of identification and alienation in the building of interpersonal antagonism. In any case, these axioms need be decomposed into a collection of simpler axioms. Only then we will be able to clearly see the specific departure points.

In a way, this has been the role played by Axioms 6, 7 and 8. Axioms 6 and 7 point towards the same direction: the emergence of polarization needs not to be around two poles only, it can be around any number of poles. Of course the smaller the number of poles—but no less than two—the higher the polarization. Axiom 6 captures exactly this as it consider the formation of an arbitrary number of poles. On this respect, the fact that both measures of bi-polarization show increases or decreases as the number of poles is even or odd is revealing. Axiom 7 captures a different perspective of the same problem. Even though the distribution is becoming closer to the extreme bi-polar distribution, the emergence of a larger number of poles may offset the polarizing effect that the new poles are closer to the extremes.

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## 7 Appendix: Other polarization measures

For the sake of completeness, we analyse two additional measures that have appeared in the literature: Zhang and Kanbur (2001) and Alesina and Spolaore (1997).

## 7.1 Zang and Kanbur

Zang and Kanbur (2001) propose a measure that is a direct translation of the intuition behind properties (ii) and (iii). Their measure is based on the family of indices of entropy as developed by Theil,  $I$ .

Theil's inequality index is

$$I(f) = \int \log \frac{\mu}{x} f(x) dx.$$

This index has well-known additive decomposability properties. Suppose that population can be partitioned into  $K$  groups on the basis of a second characteristic. Let  $\pi_i$  be the population share of the population in group  $i$ ,  $f_i(x)$  the density and  $\mu_i$  the mean of  $x$  within the  $i$ -th group,  $i = 1, \dots, K$ . As before we shall denote the discrete, simplified distribution by  $\rho$ .  $I(f)$  can be written as

$$I(f) = I^B(\rho) + I^W(f) = \sum_{i=1}^K \log \frac{\mu}{\mu_i} \pi_i + \sum_{i=1}^K \pi_i I(f_i).$$

Total inequality is thus the sum of the inequality across the groups and the population weighted within-group inequalities.

The measure of polarizarion proposed by Zhang and Kanbur is

$$P^{ZK} = \frac{I^B}{I^W} = \frac{\sum_{i=1}^K \log \frac{\mu}{\mu_i} \pi_i}{\sum_{i=1}^K \pi_i I(f_i)} .. \quad (27)$$

Since the population might be grouped on the basis of a second characteristic, the different group densities may have overlapping supports. This makes the unrestricted comparison between the performance of  $P^{ZK}$  and *the* claims in our axioms problematic. Yet, if we focus on the special case of groups with non-overlapping supports we can go some way into comparing the two measures. Specifically, in the axioms involving basic densities we shall take that each basic density constitutes a "group".

Before discussing the specific properties it is worth making a general point with respect to  $P^{ZK}$ . The Zhang and Kanbur measure is a hybrid between polarization and inequality. It has important elements of a measure of polarization because if the groups become more concentrated -and thus within-group inequality falls-  $P^{ZK}$  raises. Yet, if we leave within group inequality

unchanged and vary the size and distance across groups,  $P^{ZK}$  behaves as a pure inequality measure. In this sense,  $P^{ZK}$  is complementary to the Esteban and Ray [1994] measure. There, the population was assumed to have already been grouped and the within group distribution was left out of the analysis. The object of the Esteban and Ray (1994) the paper precisely was to provide a measure of the *polarization* —as different from the inequality— *across groups*.

Let us now check what axioms does  $P^{ZK}$  satisfy.

**Proposition 4** *The measure  $P^{ZK}$  satisfies Axioms 1 to 4 and Axiom 6 and violates Axioms 7 and 8.*

*Proof.-* Axiom 1 is concerned with a distribution consisting of one basic density which is subject to a  $\lambda$ -squeeze. Since there is one group only, the within group inequality is null. Therefore,  $P^{ZK}$  does (weakly) satisfy Axiom 1 because it will remain constant —and equal to zero— as the distribution is squeezed. In Axiom 2 there are no changes between the groups. The squeeze of the two outer groups will decrease the within-group inequality. It follows that in the scenario described in Axiom 2  $P^{ZK}$  will increase, as required. In Axiom 3 the within-group distributions are left unchanged, but the across groups inequality increases as the inner densities are pulled outwards.  $P^{ZK}$  clearly increases as demanded by Axiom 3. Finally, Axiom 4 is also satisfied since Theil's measure is normalized to total income scaling.

In Axiom 6 the across-group inequality is kept unchanged while every density becomes less unequal. This will bring  $P^{ZK}$  up, as required.

Let us now verify that  $P^{ZK}$  does not satisfy Axiom 7. This axiom considers a symmetric distribution made of four non-overlapping basic densities obtained from the same kernel and posits that the transfer of population from the inner densities towards the outer densities should have a non-monotonic effect on polarization, if the inner and outer densities are close enough. Our first observation is that since the four basic densities have been obtained from the same kernel, all will have the same within-group inequality, say  $I$  and that this will remain unchanged under the transfer of population towards the outer densities. It follows that the aggregate within-group inequality will remain unchanged too. With respect to the between-group inequality, the shifting of population towards the outer densities clearly increases the spread and thus raises between-group inequality. It follows that  $P^{ZK}$  will monotonically raise, independently of the distance between the inner and the outer densities.

Axiom 8 is not satisfied by  $P^{ZK}$  either. Flipping the distribution around the mid-point of the support does not change within group inequality, but for all non-symmetric distributions it changes the across-group inequality. Hence,  $P^{ZK}$  is not neutral to the flipping of the distribution.

At the beginning we have mentioned that this measure is a mix of polarization and inequality. Specifically, when there is no change in the within-group dispersion  $P^{ZK}$  behaves as an inequality measure. Is the Theil index over the  $n$ -group simplified distribution.

In order to make this point more precise, we shall show that  $P^{KZ}$  does not satisfy Axiom 2 in Esteban and Ray (1994) for polarization for discrete distributions. This Axiom asserts that, if a distribution consists of three spikes situated at  $0, x, x + d$  with population sizes of  $q, p, p$ , then when  $d$  is small enough relative to  $x$  and  $p$  is small enough relative to  $q$ , the fusion of the masses at  $x$  and  $x + d$  at the mid point  $x + \frac{d}{2}$  should not decrease polarization. The intuition is clear: we expect polarization to increase when the two smaller groups with little reciprocal alienation fuse to form a unite group opposing the one situated at 0. Notice that the fusion of the two groups creates a distribution that second order stochastically dominates the initial distribution. Hence, the Theil inequality measure (over the groups) will come down and so will  $P^{ZK}$ .

## 7.2 Alesina and Spolaore

Alesina and Spolaore (1997) propose a different measure of polarization. This is the median distance to the median. Formally, let  $F$  be a distribution with median  $m$ , that is  $F^{-1}(\frac{1}{2}) = m$ . The population mass at a distance  $d$  of the median is  $F(m + d) - F(m - d)$ . Then, polarization  $P^{AS}$  is implicitly defined by

$$F(m + P^{AS}) - F(m - P^{AS}) = \frac{1}{2}. \quad (28)$$

**Proposition 5** *The  $P^{AS}$  polarization measure satisfies Axioms 1 to 4 and 6 and 8, but does not satisfy Axiom 7.*

*Proof.- Let us start with Axiom 1. We have a unimodal, symmetric distribution that is subject to a  $\lambda$ -squeeze. We have already shown that for symmetric, unimodal distributions a  $\lambda$ -squeeze unequivocally concentrates mass around the median. It follows that the median distance to the median must*

come down and thus  $P^{AS}$  behaves accordingly with the statement of Axiom 1.

In Axiom 2 we have a symmetric distribution with three non-overlapping densities. The mean of the two densities on the sides are at a distance  $d$  of the global mean/median. Clearly, as long as the density at the center has a strictly positive mass it has to be that  $m - d < P^{AS} < m$ . It is plain that, in accordance with Axiom 2, a  $\lambda$ -squeeze will increase  $P^{AS}$ .

Finally, in Axiom 3 we have a symmetric distributions with four non-overlapping densities with the two inner densities being pulled towards the outer densities. It is trivial that in this case too  $P^{AS}$  will behave accordingly with the predicate of this axiom.

We find that the measure proposed by Alesina and Spolaore too satisfies all our axioms and that, therefore, it cannot belong to the class of measures based on the interplay of identification/antagonism as defined in (1).

We now test whether  $P^{AS}$  does satisfy the three new Axioms we have introduced.

Let us start with Axiom 7 in which we have a symmetric distribution with four non-overlapping densities with the same kernel. The two inner densities have means at a distance  $D$  from each other and have a population of size  $\frac{1}{2} - \delta$ , while the outer densities are at a distance  $d$  from the mean of the corresponding inner densities and have population size of  $\delta$ . Thus, increases in  $\delta$  correspond to a transfer of population from the inner densities towards the outer densities. Axiom 7 requires that if  $d$  is sufficiently small relative to  $D$  then polarization should not record a monotonic increase as  $\delta$  increases. Observe that when  $\delta = 0$  the median distance to the median is  $\frac{D}{2}$  and that as  $\delta$  increases so does (strictly)  $P^{AS}$  until it reaches its maximum  $P^{AS} = \frac{D}{2} + d$  when  $\delta = \frac{1}{2}$ . We can thus conclude that unlike the behavior required by Axiom 7,  $P^{AS}$  varies monotonically as mass is shifted from the centre towards the outer densities.

Now we turn to Axiom 6 with a uniform distribution with support  $[a, b]$  that is partitioned into  $n$  identical intervals. Then, we subject the (uniform) density of each interval to a  $\lambda$ -squeeze. We expect polarization to go up irrespective of the number of intervals  $n$ . We found that Wolfson's measure possess the odd feature that such  $\lambda$ -squeeze increases or decreases  $P^W$  as  $n$  is even or odd, respectively. The performance of  $P^{AS}$  is a bit more intricate. We start by observing that in the original distribution the mean/median is

$\frac{a+b}{2}$  and polarization is  $P^{AS} = \frac{b-a}{4}$ . Consider first the case  $n = 3$ , so that the three intervals are  $[a, a + \frac{b-a}{3}]$ ,  $[a + \frac{b-a}{3}, a + 2\frac{b-a}{3}]$ , and  $[a + 2\frac{b-a}{3}, b]$ . The points at a distance  $\frac{b-a}{4}$  from the median belong to the first and third interval and are located between the mean of the interval and the global mean/median. It follows that a  $\lambda$ -squeeze will pull this population away from the global mean (and towards the local mean) thus making  $P^{AS}$  go up as required. It can be easily seen that the argument extends to any number of intervals as long as  $n$  is odd.

Let us now address the case of an even number intervals with a population  $\frac{1}{n}$ . Notice that now the mean/median is located at the frontier between the intervals  $\frac{n}{2}$  and  $\frac{n}{2} + 1$ . In order to identify the median distance to the median we depart symmetrically by a distance  $d$  from the median  $m$  until we reach  $d^*$  so that the total population mass in between  $m - d^*$  and  $m + d^*$  is  $\frac{1}{2}$ . Whenever  $n$  is multiple of 4, these two points too will be located at the frontier between two adjacent intervals; otherwise, they will coincide with the (conditional) mean of the appropriate densities. When performing a  $\lambda$ -squeeze of the  $n$  densities the support of all the densities will shrink towards the corresponding (conditional) mean. Thus, the location of the overall median and (for  $n$  multiple of 4) of the median distance to the median is not unequivocal. One way to handle this case is to consider that the  $n$ -interval case is the limit of performing the partition, assigning an  $\epsilon$  mass of population at each of these critical points and let  $\epsilon \rightarrow 0$ . It is immediate to see that any  $\lambda$ -squeeze over an even number of adjacent intervals will leave  $P^{AS}$  unchanged. It follows that for  $n = 2$ , for instance, the Alesina-Spolaore measure  $P^{AS}$  considers equally polarized a uniform distribution over  $[a, b]$  and a symmetric two-spike distribution at a distance  $\frac{b-a}{2}$  of each other.

Summing up our analysis of the performance of  $P^{AS}$  in the scenario of Axiom 6, we find that it behaves strictly accordingly with Axiom 6 if the number of intervals is odd, while only weakly when it is even.

Let us finally show that  $P^{AS}$  does satisfy Axiom 8. We start with density  $f$  with support  $[a, b]$ . The degree of polarization is the number  $P_f^{AS}$  satisfying

$$\int_{m_f - P_f^{AS}}^{m_f + P_f^{AS}} f(x) dx = \frac{1}{2} \text{ with } F(m_f) = \frac{1}{2}.$$

Consider now the density  $g$  such that  $g(x) = f(2M - x)$  with  $M = \frac{a+b}{2}$ . Polarization will now be  $P_g^{AS}$  satisfying,

$$\int_{m_g - P_g^{AS}}^{m_g + P_g^{AS}} g(x) dx = \frac{1}{2} \text{ with } G(m_g) = \frac{1}{2}.$$

Now, we have that

$$\begin{aligned} \frac{1}{2} &= \int_{m_g - P_g^{AS}}^{m_g + P_g^{AS}} g(x) dx = \int_{m_g - P_g^{AS}}^{m_g + P_g^{AS}} f(2M - x) dx = \\ (\text{making } z &= 2M - x) = \int_{2M - m_g - P_g^{AS}}^{2M - m_g + P_g^{AS}} f(z) dz = \int_{m_f - P_g^{AS}}^{m_f + P_g^{AS}} f(x) dx. \end{aligned}$$

We can thus conclude that  $P_f^{AS} = P_g^{AS}$ .

Concerning the Alesina-Spolaore measure we have shown that it satisfies Axioms 1 to 4, 6 and 8. An additional unappealing feature of  $P^{AS}$  is that for any two-spike distribution  $(p, 1 - p)$  with  $p \neq \frac{1}{2}$ , we have that  $P^{AS} = 0$ , irrespective of the distance  $D$  between the two spikes. Yet, one would intuitively consider this scenario to correspond to a highly polarized distribution.

The fact that such an inappropriate measure satisfies most of the axioms seems to suggest that Axioms 6, 7 and 8 are far from significantly capturing the essential features of Axiom 5.